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Ellipsoidal Harmonic Analysis

G. H. Darwin

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XII. *Ellipsoidal Harmonic Analysis.*

By G. H. Darwin, *F.R.S., Plumian Professor and Fellow of Trinity College, in the University of Cambridge.*

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Introduction.

Lamé's functions or ellipsoidal harmonics have been successfully used in many investigations, but the form in which they have been presented has always been such as to render numerical calculation so difficult as to be practically impossible. The object of the present investigation is to remove this imperfection in the method. I believe that I have now reduced these functions to such a form that numerical results will be accessible, although by the nature of the case the arithmetic will necessarily remain tedious.

Throughout my work on ellipsoidal harmonics I have enjoyed the immense advantage of frequent discussions with Mr. E. W. Hobson. He has helped me freely from his great store of knowledge, and beginning, as I did, in almost complete ignorance of the subject, I could hardly have brought my attempt to a successful issue without his advice. In many cases the help derived from him has been of immense value, even where it is not possible to indicate a specific point as due to him. In other cases he has put me in the way of giving succinct proofs of propositions which I had only proved by clumsy and tedious methods, or where I merely felt sure of the truth of a result without rigorous proof. In particular, I should have been quite unable to carry out the investigation of § 19, unless he had shown me how the needed series were to be determined.

My original object in attacking this problem was the hope of being thereby enabled to obtain exact numerical results with respect to M. Poincaré's pear-shaped figure of equilibrium of a mass of liquid in rotation.* But I soon found that a partial investigation with one particular point in view was impracticable, and I was thus led on little by little to cover the whole field, in as far as it was necessary to do so for the purpose of practical application. This paper has then grown to such considerable dimensions that it seemed best that it should stand by itself, and that the discussion of the specific problem should be deferred.

A paper of this kind is hardly read even by the mathematician, unless he happens to be working at a cognate subject. It appears therefore to be useful to present a summary, which shall render it possible for the mathematical reader to understand the nature of the method and results, without having to pick it out from a long and complex train of analysis. Such a summary is given in Part III.

PART I.

Formation of the Functions.

§ 1. *The Principles of Ellipsoidal Harmonic Analysis.*

The basis of this method of analysis is expounded in various works on the subject. I begin with a statement of results in my own notation.

* A paper giving the required result will be presented to the Society in the autumn.—[*July* 2, 1901.]

If u_1^2, u_2^2, u_3^2 denote the three roots of the cubic

$$\frac{x^2}{a^2+u^2}+\frac{y^2}{b^2+u^2}+\frac{z^2}{c^2+u^2}=1,$$

it may be proved that

$$x^2=\frac{(a^2+u_1^2)(a^2+u_2^2)(a^2+u_3^2)}{(b^2-a^2)(c^2-a^2)},$$

and y^2, z^2 may be written down by cyclical changes.

If for brevity we write

$$A_n^2=u_n^2+a^2,\ B_n^2=u_n^2+b^2,\ C_n^2=u_n^2+c^2,\ (n=1,2,3),$$

LAPLACE's equation becomes

$$(u_2^2-u_3^2)\left(A_1B_1C_1\frac{d}{u_1du_1}\right)^2V_i+(u_3^2-u_1^2)\left(A_2B_2C_2\frac{d}{u_2du_2}\right)^2V_i$$
$$+(u_1^2-u_2^2)\left(A_3B_3C_3\frac{d}{u_3du_3}\right)V_i=0.$$

The solution is $$V_i=U_1U_2U_3,$$

where U_1, U_2, U_3 are functions of u_1, u_2, u_3 respectively, and satisfy

$$\left(A_1B_1C_1\frac{d}{u_1du_1}\right)^2U_1=[i(i+1)u_1^2+\kappa^2]U_1,$$

and two other equations with suffixes 2 and 3, involving the same κ, a constant, and the same i, a positive integer.

If a, b, c are in ascending order of magnitude we may suppose u_1^2 to lie between $-a^2$ and ∞, u_2^2 between $-c^2$ and $-b^2$, and u_3^2 between $-b^2$ and $-a^2$.

If s_1, s_2, s_3 denote the three orthogonal arcs formed by the intersections of the three orthogonal quadrics,

$$\left(\frac{ds_1}{u_1du_1}\right)^2=\frac{(u_2^2-u_1^2)(u_3^2-u_1^2)}{A_1^2B_1^2C_1^2}$$

and two other equations found by cyclical changes of suffixes.

§ 2. *Notation; limits of β so as to represent all Ellipsoids.*

I now change the notation, and let the three roots be defined thus:—

$$u_1^2=k^2\nu^2,$$
$$u_2^2=k^2\mu^2,$$
$$u_3^2=k^2\frac{1-\beta\cos 2\phi}{1-\beta},$$

where ν ranges from ∞ to 0, μ between ± 1, ϕ between 0 and 2π.

Let the axes of the fundamental ellipsoid of reference be

$$a^2 = -k^2\frac{1+\beta}{1-\beta},$$
$$b^2 = -k^2,$$
$$c^2 = 0.$$

The ellipsoid defined by ν has its three axes a, b, c given by

$$a^2 = k^2\left(\nu^2 - \frac{1+\beta}{1-\beta}\right),\ b^2 = k^2(\nu^2 - 1),\ c^2 = k^2\nu^2,\ (a < b < c).$$

This mode of defining the axis is such as to indicate the relationship to the prolate ellipsoid $a = b < c$. But another hypothesis may be made which will bring the axes into relationship with those of the oblate ellipsoid $a = b > c$; for if we take a new k, numerically equal to the old one but imaginary, and replace ν^2 by $-\zeta^2$, we have

$$a^2 = k^2\left(\zeta^2 + \frac{1+\beta}{1-\beta}\right),\ \beta^2 = k^2(\zeta^2 + 1),\ c^2 = k^2\zeta^2,\ (a > b > c).$$

If β be made to range from 0 to ∞, all possible ellipsoids are comprised in either of these types. It will, however, now be shown that, by a proper choice of type, all ellipsoids may be included with the range of β from 0 to $\frac{1}{3}$.

Let us suppose the axes to be expressed in three forms, as follows :—

$$\begin{aligned}
&\quad(1.) && \quad(2.) && \quad(3.)\\
a^2 &= k^2\left(\nu^2 - \frac{1+\beta}{1-\beta}\right) &&= k_1^2\zeta_1^2 &&= k_2^2\nu_2^2,\\
b^2 &= k^2(\nu^2 - 1) &&= k_1^2(\zeta_1^2 + 1) &&= k_2^2\left(\nu_2^2 - \frac{1+\beta_2}{1-\beta_2}\right),\\
c^2 &= k^2\nu^2 &&= k_1^2\left(\zeta_1^2 + \frac{1+\beta_1}{1-\beta_1}\right) &&= k_2^2(\nu_2^2 - 1).
\end{aligned}$$

Then we have

$$b^2 - a^2 = \frac{2k^2\beta}{1-\beta} = k_1^2 = -k_2^2\frac{1+\beta_2}{1-\beta_2}$$
$$c^2 - b^2 = k^2 = \frac{2k_1^2\beta_1}{1-\beta_1} = \frac{2k_2^2\beta_2}{1-\beta_2}.$$

Therefore
$$\frac{b^2-a^2}{c^2-b^2} = \frac{2\beta}{1-\beta} = \frac{1-\beta_1}{2\beta_1} = -\frac{1+\beta_2}{2\beta_2},$$
$$\frac{c^2-a^2}{c^2-b^2} = \frac{1+\beta}{1-\beta} = \frac{1+\beta_1}{2\beta_1} = -\frac{1-\beta_2}{2\beta_2}$$

And
$$\frac{b^2-a^2}{2c^2-a^2-b^2} = \beta = \frac{1-\beta_1}{1+3\beta_1} = \frac{1+\beta_2}{1-3\beta}$$

Now let β increase from 0 to ∞.

As β passes from 0 to $\frac{1}{3}$, form (1) is appropriate.

As β passes from $\frac{1}{3}$ to 1, β_1 decreases from $\frac{1}{3}$ to 0, so that form (2) is appropriate.

Lastly, as β passes from 1 to ∞, β_2 increases from 0 to $\frac{1}{3}$, so that form (3) is appropriate.

But we might equally well have written forms (1) and (3) so as to involve ζ, and form (2) so as to involve ν, and it follows that all possible ellipsoids are comprised in the range of β from 0 to $\frac{1}{3}$, provided that the type be appropriately chosen.

The developments in this paper are made in powers of β. It will, therefore, be well to show that there is a class of ellipsoids, analogous to ellipsoids of revolution, which might form the basis of developments similar to those carried out below.

Ellipsoids of revolution are defined by the condition

$$a^2 - c^2 = b^2 - c^2, \text{ or } a^2 = b^2.$$

In the class to which I refer

$$a^2 - c^2 = c^2 - b^2, \text{ or } c^2 = \tfrac{1}{2}(a^2 + b^2).$$

Ellipsoids of this kind are given by $\beta = \beta_1 = -\beta_2 = \frac{1}{3}$; for in this case $b^2 = \frac{1}{2}(a^2 + c^2)$. They are also given by

$$\beta = \infty, \; -\beta_1 = \beta_2 = \tfrac{1}{3}; \text{ for then } c^2 = \tfrac{1}{2}(a^2 + b^2).$$

Hence if we only allow β to range from 0 to $\frac{1}{3}$, $\beta = 0$ corresponds with ellipsoids of revolution, to which spheroidal harmonic analysis is applicable; and $\beta = \frac{1}{3}$ corresponds with this new class for which the corresponding analysis has not yet been worked out.

We shall see below that the solid harmonic for this case where $\beta = \frac{1}{3}$ will be of the form $\mathrm{B}(\nu)\,\mathrm{B}(\mu)\,\mathrm{E}(\phi)$, where B and E satisfy the equations

$$(\nu^2 + 1)(\nu^2 - 1)\frac{d^2\mathrm{B}}{d\nu^2} + 2\nu^3\frac{d\mathrm{B}}{d\nu} - i(i+1)\nu^2\mathrm{B} + s^2\mathrm{B} = 0,$$

$$\cos 2\phi\,\frac{d^2\mathrm{E}}{d\phi^2} - \sin 2\phi\,\frac{d\mathrm{E}}{d\phi} + i(i+1)\,\mathrm{E}\cos 2\phi - s^2\mathrm{E} = 0.$$

I am not clear whether or not it would be advisable to proceed *ab initio* from these equations, but at any rate I shall show hereafter how the B- and E-functions may be determined from the analysis of the present paper with any degree of accuracy desirable.

If it were proposed to use the functions corresponding to $\beta = \frac{1}{3}$ as a basis for the development of general ellipsoidal harmonics, we should have to assume

$$a^2 = k'^2\nu'^2,\ b^2 = k'^2(\nu'^2 - 1),\ c^2 = k'^2\left(\nu'^2 - \frac{2}{1-\eta}\right);$$

or else
$$a^2 = k'^2\zeta'^2,\ b^2 = k'^2(\zeta'^2 + 1),\ c^2 = k'^2\left(\zeta'^2 + \frac{2}{1-\eta}\right).$$

The developments would then proceed by powers of η.

In order to discover what is the greatest value of η which must be used so as to comprise all ellipsoids, when we proceed from both bases of development, a comparison must be made between this assumption and the previous one. Suppose in fact that

$$a^2 = k^2\left(\nu^2 - \frac{1+\beta}{1-\beta}\right) = k'^2\zeta'^2;\ b^2 = k^2(\nu^2 - 1) = k'^2(\zeta'^2 + 1);$$

$$c^2 = k^2\nu^2 = k^2\left(\zeta'^2 + \frac{2}{1-\eta}\right).$$

Then
$$b^2 - a^2 = \frac{2k^2\beta}{1-\beta} = k'^2;\ c^2 - b^2 = k^2 = k'^2\frac{1+\eta}{1-\eta},$$

and therefore
$$\frac{2\beta}{1-\beta} = \frac{1-\eta}{1+\eta},\ \text{or}\ \eta = \frac{1-3\beta}{1+\beta}.$$

When η and β are both equally great, they must each equal the positive root of $\beta = \frac{1-3\beta}{1+\beta}$. This root is $\sqrt{5} - 2$ or $\cdot 236$. Thus the greatest values will be

$$\beta = \eta = \frac{1}{\sqrt{5}+2} = \frac{1}{4\cdot 236}.$$

In this case $\eta^2 = \beta^2 = \frac{1}{18}$ very nearly, whereas when $\beta = \frac{1}{3}$, $\beta^2 = \frac{1}{9}$. Thus if the developments were to stop with β^2 we should double the accuracy of the result. However, I do not at present propose to carry out the process suggested.

§ 3. *The Differential Equations.*

We now put $u_1^2 = k^2\nu^2$, $u_2^2 = k^2\mu^2$, $u_3^2 = k^2\frac{1-\beta\cos 2\phi}{1-\beta}$;

$$a^2 = -k^2\frac{1+\beta}{1-\beta},\qquad b^2 = -k^2,\qquad c^2 = 0;$$

and find from the formulæ of § 1,

$$\left.\begin{aligned}\frac{x^2}{k^2} &= -\frac{1-\beta}{1+\beta}\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)\left(\mu^2 - \frac{1+\beta}{1-\beta}\right)\cos^2\phi,\\ \frac{y^2}{k^2} &= -(\nu^2-1)(\mu^2-1)\sin^2\phi,\\ \frac{z^2}{k^2} &= \nu^2\mu^2\frac{1-\beta\cos 2\phi}{1+\beta}\end{aligned}\right\}\quad \ldots\ (1).$$

It will be observed that y is independent of β, and that it has the same form as in spheroidal harmonic analysis when β vanishes. Since μ^2 is less than 1 and ν^2 greater than $\frac{1+\beta}{1-\beta}$, x and y are real.

In all the earlier portion of this paper I always write $\mu^2 - 1$ and not $1 - \mu^2$, so as to maintain perfect symmetry with respect to ν and μ.

We now have

$$A_1^2 = k^2\left(\nu^2 - \frac{1+\beta}{1-\beta}\right), \qquad B_1^2 = k^2(\nu^2 - 1), \qquad C_1^2 = k^2\nu^2;$$

$$A_2^2 = k^2\left(\mu^2 - \frac{1+\beta}{1-\beta}\right), \qquad B_2^2 = k^2(\mu^2 - 1), \qquad C_2^2 = k^2\mu^2;$$

$$A_3^2 = \frac{-2k^2\beta\cos^2\phi}{1-\beta}, \qquad B_3^2 = \frac{2k^2\beta\sin^2\phi}{1-\beta}, \qquad C_3^2 = k^2\frac{1-\beta\cos 2\phi}{1-\beta}.$$

Let us denote the differential operators involved in our equations, thus :—

$$D_1 = (1-\beta)^{\frac{1}{2}}\frac{A_1B_1C_1}{ku_1}\frac{d}{du_1}, \qquad D_2 = (1-\beta)^{\frac{1}{2}}\frac{A_2B_2C_2}{ku_2}\frac{d}{du_2},$$
$$D_3 = -\sqrt{-1}\,.\,(1-\beta)^{\frac{1}{2}}\frac{A_3B_3C_3}{ku_3}\frac{d}{du_3}.$$

Then

$$\left.\begin{aligned} D_1 &= (1-\beta)^{\frac{1}{2}}\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}(\nu^2-1)^{\frac{1}{2}}\frac{d}{d\nu}, \\ D_2 &= (1-\beta)^{\frac{1}{2}}\left(\mu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}(\mu^2-1)^{\frac{1}{2}}\frac{d}{d\mu}, \\ D_3 &= (1-\beta\cos 2\phi)^{\frac{1}{2}}\frac{d}{d\phi} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2).$$

$$A_1B_1C_1\frac{d}{u_1du_1} = \frac{kD_1}{(1-\beta)^{\frac{1}{2}}}, \qquad A_2B_2C_2\frac{d}{u_2du_2} = \frac{kD_2}{(1-\beta)^{\frac{1}{2}}},$$
$$A_3B_3C_3\frac{d}{u_3du_3} = \sqrt{-1}\,.\,\frac{kD_3}{(1-\beta)^{\frac{1}{2}}}.$$

Hence our differential equations are

$$\frac{D_1^2U_1}{1-\beta} = \left[i(i+1)\nu^2 + \frac{\kappa^2}{k^2}\right]U_1, \text{ a similar equation with suffix 2, and}$$
$$\frac{-D_3^2U_3}{1-\beta} = \left[i(i+1)\frac{1-\beta\cos 2\phi}{1-\beta} + \frac{\kappa^2}{k^2}\right]U_3.$$

Let us replace κ^2 by another constant such that

$$\left(i(i+1)\nu^2 + \frac{\kappa^2}{k^2}\right)(1-\beta) = i(i+1)[\nu^2(1-\beta) - 1] + s^2 - \beta\sigma,$$

so that
$$\frac{\kappa^2}{k^2} = -\frac{i(i+1) - s^2 + \beta\sigma}{1-\beta}$$

In this formula s is a constant integer and σ a constant to be determined.

Our equations are now

$$\left.\begin{array}{l} [D_1^2 - i(i+1)[\nu^2(1-\beta)-1] - s^2 + \beta\sigma]\,U_1 = 0, \\ \text{a similar equation for } \mu \\ \text{and} \quad [D_3^2 - i(i+1)\beta\cos 2\phi + s^2 - \beta\sigma]\,U_3 = 0 \end{array}\right\} \quad . \quad . \quad (3).$$

And LAPLACE'S equation is

$$\left[\left(\mu^2 - \frac{1-\beta\cos 2\phi}{1-\beta}\right)D_1^2 + \left(\frac{1-\beta\cos 2\phi}{1-\beta} - \nu^2\right)D_2^2 - (\nu^2-\mu^2)\,D_3^2\right]U_1U_2U_3 = 0. \quad . \quad . \quad . \quad . \quad (4).$$

LAPLACE'S operator ∇^2 is equal to the differential operator in (4), divided by $-k^2(\nu^2-\mu^2)\left(\nu^2 - \frac{1-\beta\cos 2\phi}{1-\beta}\right)\left(\frac{1-\beta\cos 2\phi}{1-\beta} - \mu^2\right)$.

It is well known that in spheroidal harmonic analysis there are two kinds of functions of ν and μ which satisfy the differential equation, and they are usually denoted P_i^s, Q_i^s. The Q-functions of the variable μ have no significance, so that virtually there are P- and Q-functions of ν, but only P-functions of μ. The like is true in the present case, however, with the additional complication that each of the functions may assume one of two alternative forms. I adopt a parallel notation and write for U_1 and U_2 either $\mathfrak{P}_i^s$, $\mathfrak{Q}_i^s$, or P_i^s, Q_i^s, as the case may be. Since ν and μ enter in the first two equations in exactly the same way, we need only consider one of them, and we may usually write simply (for example) $\mathfrak{P}_i^s$ where the full notation would be $\mathfrak{P}_i^s(\nu \text{ or } \mu)$. In the early part of the investigation I shall only refer to the P-functions, and the Q-functions will be considered later.

In spheroidal harmonic analysis the third function is a cosine or sine of $s\phi$. So here also we find functions of two kinds associated with cosines and sines, which I shall denote $\mathfrak{C}_i^s$, $\mathfrak{S}_i^s$, C_i^s, S_i^s, the variable ϕ being understood.

Throughout the greater part of this paper the functions will be of degree denoted by i, and it seems useless to print the subscript i hundreds of times. I shall accordingly drop the subscript i except where it shall be necessary or advisable to retain it; for example, $\mathfrak{P}^s$ will be the abridged notation for $\mathfrak{P}_i^s(\nu)$.

The operators involved in the differential equations (3) will occur so frequently that an abridged notation seems justifiable. I therefore write

$$\left.\begin{array}{l} \psi_s = D_1^2 - i(i+1)[\nu^2(1-\beta)-1] - s^2 + \beta\sigma, \\ \chi_s = D_3^2 - i(i+1)\beta\cos 2\phi + s^2 - \beta\sigma, \\ \text{where} \quad D_1 = (1-\beta)^{\frac{1}{2}}\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}(\nu^2-1)^{\frac{1}{2}}\frac{d}{d\nu}, \\ D_3 = (1-\beta\cos 2\phi)^{\frac{1}{2}}\frac{d}{d\phi} \end{array}\right\} \quad . \quad . \quad . \quad . \quad (5).$$

The equations are then

$$\left.\begin{aligned}\psi_s(\mathfrak{P}^s \text{ or } \mathsf{P}^s) &= 0\,,\\ \chi_s(\mathfrak{C}^s \text{ or } \mathfrak{S}^s \text{ or } \mathsf{C}^s \text{ or } \mathsf{S}^s) &= 0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (5).$$

§ 4. *The Forms of the Functions.*

It is well known that the function U_i is a linear function of u_1 of degree i made up in one of the eight following ways :—

1. When i is even, a linear function of u_1^2 of degree $\frac{1}{2}i$.

2, 3, 4. When i is odd, a linear function of u_1^2 of degree $\frac{1}{2}(i-1)$, multiplied by A_1, or B_1, or C_1.

5, 6, 7. When i is even, a linear function of u_1^2 of degree $\frac{1}{2}(i-2)$, multiplied by B_1C_1, or C_1A_1, or A_1B_1.

8. When i is odd, a linear function of u_1^2 of degree $\frac{1}{2}(i-3)$, multiplied by $A_1B_1C_1$.

These eight classes may be conveniently specified by the initials O, A, B, C, BC, CA, AB, ABC, but it is better to rearrange them according as they are associated with the evenness or oddness of i and s, and with the cosine or sine functions. This new grouping may be defined by a shorthand notation involving the initials E, O and C or S, which shall denote successively the evenness or oddness of i and s, and cosine or sine.

We shall see below that this arrangement is as follows :—

O or EEC ; i even, s even, cosine.

AB or EES ; i even, s even, sine.

A or OOC ; i odd, s odd, cosine.

B or OOS ; i odd, s odd, sine.

C or OEC ; i odd, s even, cosine.

ABC or OES ; i odd, s even, sine.

CA or EOC ; i even, s odd, cosine.

CB or EOS ; i even, s odd, sine.

Since the several functions are linear in u_1^2, they are in the new notation functions of ν^2 or μ^2, or of ν^2-1 and μ^2-1.

Hence $\mathfrak{P}^s(\nu)$ and $\mathsf{P}^s(\nu)$ involve linear functions of ν^2-1 of various degrees multiplied by various factors ; and the same is true of the functions of μ.

In the case of the third root the linear function of powers of $\cos 2\phi$ may be replaced by a series of cosines of even multiples of ϕ. Further, in forming the $\mathfrak{C}$, $\mathfrak{S}$, C, S functions we may regard A_3 as being $\cos\phi$, B_3 as $\sin\phi$, and C_3 as $(1-\beta\cos 2\phi)^{\frac{1}{2}}$, since this only amounts to dropping constant factors which may be deemed to be included in the, as yet, undetermined coefficients of the several series.

I will now consider in detail the forms of the several P-functions of ν (those for μ following by symmetry), and at the same time indicate more precisely the nature of the notation adopted.

In the following series, indicated by Σ, the variable t is supposed *to proceed from the lower to the upper limit by* 2 *at a time*. The reader will be able to perceive the manner of the formation of the functions when he bears in mind that

$$A_1 = k\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}, \quad B_1 = k(\nu^2 - 1)^{\frac{1}{2}}, \quad C_1 = k\nu.$$

Type O or EEC; $\quad \mathfrak{P}^s = \sum_0^i \alpha_t (\nu^2 - 1)^{\frac{1}{2}t}$.

Type AB or EES; $\quad \mathsf{P}^s = \sum_2^i \alpha_t (\nu^2 - 1)^{\frac{1}{2}(t-1)} \left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}$.

Type A or OOC; $\quad \mathsf{P}^s = \sum_1^i \alpha_t (\nu^2 - 1)^{\frac{1}{2}(t-1)} \left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}$.

Type B or OOS; $\quad \mathfrak{P}^s = \sum_1^i \alpha_t (\nu^2 - 1)^{\frac{1}{2}t}$.

Type C or OEC; $\quad \mathfrak{P}^s = \sum_1^i \alpha_t \nu (\nu^2 - 1)^{\frac{1}{2}(t-1)}$.

Type ABC or OES; $\quad \mathsf{P}^s = \sum_3^i \alpha_t \nu (\nu^2 - 1)^{\frac{1}{2}(t-2)} \left(\nu^2 - \frac{1+\beta}{1-\beta}\right)$.

Type CA or EOC; $\quad \mathsf{P}^s = \sum_2^i \alpha_t \nu (\nu^2 - 1)^{\frac{1}{2}(t-2)} \left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}$.

Type CB or EOS; $\quad \mathfrak{P}^s = \sum_2^i \alpha_t \nu (\nu^2 - 1)^{\frac{1}{2}(t-1)}$.

Observe that P is always associated with $\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}$, and that, each form being repeated twice, there are two forms of function of each kind. Moreover, a cosine and a sine function are always associated with different kinds. It is obvious that the $\mathfrak{P}$-functions are expressible in terms of the ordinary P-functions of spherical harmonic analysis, and that if we take out the factor $\left(\frac{\nu^2 - \frac{1+\beta}{1-\beta}}{\nu^2 - 1}\right)^{\frac{1}{2}}$ the P-functions are similarly expressible. This factor will occur so frequently that I write

$$\Omega(\nu) = \left(\frac{\nu^2 - \frac{1+\beta}{1-\beta}}{\nu^2 - 1}\right)^{\frac{1}{2}},$$

and as elsewhere commonly put Ω to denote $\Omega(\nu)$.

We assume then the following forms for the functions :—

For EEC, OEC, OOS, EOS

$$\mathfrak{P}^s = q_s \mathrm{P}^s + \Sigma\beta^n q_{s-2n}\mathrm{P}^{s-2n} + \Sigma\beta^n q_{s+2n}\mathrm{P}^{s+2n}.$$

For EES, OES, OOC, EOC

$$\mathsf{P}^s = \Omega\{q'_s \mathrm{P}^s + \Sigma\beta^n q'_{s-2n}\mathrm{P}^{s-2n} + \Sigma\beta^n q'_{s+2n}\mathrm{P}^{s+2n}\} \quad . \quad . \quad . \quad . \quad . \quad (6).$$

In these series n proceeds by intervals of one at a time, beginning from a lower limit of unity. In both forms the upper limit of the first Σ is $\frac{1}{2}s$ or $\frac{1}{2}(s-1)$ according as s is even or odd; and the upper limit of the second Σ is $\frac{1}{2}(i-s)$ or $\frac{1}{2}(i-s-1)$ according as i and s agree or do not agree in evenness or in oddness.

The factor Ω contains $(\nu^2-1)^{\frac{1}{2}}$ in the denominator, but P^s does not become infinite when $\nu = \pm 1$, because when s is not zero P^s is divisible by $(\nu^2-1)^{\frac{1}{2}}$ and we shall see that q'_0 is zero.* When s is zero there is no function of the P type.

It may be noted that the limits of the series are such that neither q nor q' can ever have a negative suffix.

We shall ultimately make q_s and q'_s equal to unity, and this will be justifiable because there must be one arbitrary constant.

We have now to consider the forms of the cosine and sine functions. They may be derived at once from the preceding results, for we have only to read $(\nu^2-1)^{\frac{1}{2}t}$ as $\cos t\phi$ where t is even; $(\nu^2-1)^{\frac{1}{2}}$ as $\sin\phi$, $\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}$ as $\cos\phi$, and ν as $(1-\beta\cos 2\phi)^{\frac{1}{2}}$.

The factor $(1-\beta\cos 2\phi)^{\frac{1}{2}}$ will occur frequently, and I write

$$\Phi(\phi) = (1-\beta\cos 2\phi)^{\frac{1}{2}},$$

and as elsewhere I commonly write Φ to denote $\Phi(\phi)$.

The following are the results :—

Type O or EEC ; $\quad \mathfrak{C}^s = \sum_0^i \gamma_t \cos t\phi.$

Type AB or EES ; $\quad \mathfrak{S}^s = \sum_2^i \gamma_t \sin t\phi.$

It is clear that we may equally well regard the lower limit in the latter as zero.

Type A, or OOC ; each term is of type $\cos(t-1)\phi\cos\phi$ or $\cos(t-2)\phi + \cos t\phi$.

Hence $\quad \mathfrak{C}^s = \sum_1^i \gamma_t \cos t\phi.$

* This also follows from the fact that the series for P^s begins with $\Omega\alpha_2(\nu^2-1)$ in the case of EES, and with $\Omega\alpha_3\nu(\nu^2-1)$ in the case of OES. Thus in the former case there is no term $\Omega\alpha_0$ and in the latter no term $\Omega\alpha_1\nu$

Type B, or OOS; since we now have $\cos(t-1)\phi\sin\phi$,

$$\mathfrak{S}^s = \sum_1^i \gamma_t \sin t\phi .$$

Type C, or OEC; $\quad \mathbf{C}^s = \Phi \sum_1^i \gamma_t \cos(t-1)\phi .$

Type ABC, or OES; each term is of type $\Phi\cos(t-1)\phi\sin\phi\cos\phi$, which gives $[\sin(t+1)\phi - \sin(t-3)\phi]\,\Phi$. Hence

$$\mathbf{S}^s = \Phi \sum_3^i \gamma_t \sin(t-1)\phi .$$

It is clear that we may equally well regard the lower limit as unity.

Type CA, or EOC; each term is of type $\Phi\cos(t-2)\phi\cos\phi$. Hence

$$\mathbf{C}^s = \Phi \sum_2^i \gamma_t \cos(t-1)\phi .$$

Type CB, or EOS; each term is of type $\Phi\cos(t-2)\phi\sin\phi$. Hence

$$\mathbf{S}^s = \Phi \sum_2^i \gamma_t \sin(t-1)\phi .$$

When i and s agree as to evenness or oddness we have the forms independent of Φ, when they differ in this respect the factor Φ occurs.

Therefore (in alternative form) for EEC, EES, OOC, OOS

$$\begin{Bmatrix}\mathfrak{C}^s\\ \mathfrak{S}^s\end{Bmatrix} = p_s \begin{Bmatrix}\cos\\ \sin\end{Bmatrix} s\phi + \Sigma\beta^n p_{s-2n}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(s-2n)\phi + \Sigma\beta^n p_{s+2n}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(s+2n)\phi$$

and for OEC, OES, EOC, EOS

$$\begin{Bmatrix}\mathbf{C}^s\\ \mathbf{S}^s\end{Bmatrix} = \Phi\left[p'_s \begin{Bmatrix}\cos\\ \sin\end{Bmatrix} s\phi + \Sigma\beta^n p'_{s-2n}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(s-2n)\phi + \Sigma\beta^n p'_{s+2n}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(s+2n)\phi\right] \quad . \quad (7).$$

In these series n proceeds by intervals of one at a time, beginning with unity. In both forms the upper limit of the first Σ is $\frac{1}{2}s$ or $\frac{1}{2}(s-1)$ according as s is even or odd. In the first form the upper limit of the second Σ is $\frac{1}{2}(i-s)$, and in the second form it is $\frac{1}{2}(i-s-1)$.

We shall ultimately put p_s and p'_s, which may be regarded as arbitrary constants, equal to unity.

§ 5. *Preparation for determination of the Functions.*

In order to determine the coefficients q, q', p, p' and σ, we have to substitute these assumed forms in the differential equations.

Where the functions involve Ω and Φ as factors, the forms already given for the differential equations are perhaps the most convenient, but in the other cases a reduction seems desirable.

By considering the forms of D_1 and D_3 in (3) it is easy to show that

$$\psi_s = \left[(\nu^2 - 1)\frac{d}{d\nu}\right]^2 - i(i+1)(\nu^2-1) - s^2$$
$$- \beta\left[(\nu^2-1)(\nu^2+1)\frac{d^2}{d\nu^2} + 2\nu^3\frac{d}{d\nu} - i(i+1)\nu^2 - \sigma\right] \quad . \quad . \quad (8),$$

$$\chi_s = \frac{d^2}{d\phi^2} + s^2 - \beta\left[\cos 2\phi\frac{d^2}{d\phi^2} - \sin 2\phi\frac{d}{d\phi} + i(i+1)\cos 2\phi + \sigma\right] \quad . \quad . \quad (9).$$

By making β vanish we reduce these operators to the forms appropriate to spheroidal harmonic analysis. By making β infinite we obtain the differential equations specified in § 2 as appropriate to ellipsoids of the class $c^2 = \frac{1}{2}(a^2+b^2)$.

It is now necessary to perform the operation ψ_s on typical terms P^t and ΩP^t, and χ_s on typical terms $\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} t\phi$ and $\Phi\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} t\phi$.

(α.) *To find* $\psi_s(P^t)$.

The form (8) for ψ_s is here convenient.

It is clear that

$$\left\{\left[(\nu^2-1)\frac{d}{d\nu}\right]^2 - i(i+1)(\nu^2-1) - s^2\right\} P^t = (t^2 - s^2) P^t,$$

because P^t is the solution of the differential equation found by erasing the term $-s^2P^t$ from each side.

Again we have from the same differential equation

$$(\nu^2-1)\frac{d^2}{d\nu^2}P^t = -2\nu\frac{dP^t}{d\nu} + i(i+1)P^t + \frac{t^2}{\nu^2-1}P^t.$$

It may be noted in passing that this is equally true when the subject of operation is Q^t, the function of the other form.

Therefore

$$\left[(\nu^2-1)(\nu^2+1)\frac{d^2}{d\nu^2} + 2\nu^3\frac{d}{d\nu} - i(i+1)\nu^2 - \sigma\right]P^t$$
$$= \left[-2\nu\frac{d}{d\nu} + i(i+1) + t^2\frac{\nu^2+1}{\nu^2-1} - \sigma\right]P^t.$$

Hence

$$\psi_s(P^t) = (t^2-s^2)P^t - \beta\left\{-2\nu\frac{d}{d\nu} + i(i+1) + t^2\frac{\nu^2+1}{\nu^2-1} - \sigma\right\}P^t.$$

We have now to eliminate $\nu \frac{d\mathrm{P}^t}{d\nu}$ and $\frac{\nu^2 + 1}{\nu^2 - 1}\mathrm{P}^t$.

It is known that
$$\mathrm{P} = \frac{1}{2^i . i\,!}\left(\frac{d}{d\nu}\right)^i (\nu^2 - 1)^i,$$
and
$$\mathrm{P}^t = (\nu^2 - 1)^{\frac{1}{2}t}\left(\frac{d}{d\nu}\right)^t \mathrm{P}.$$

The differential equation satisfied by P^t involves t in the form t^2. Hence $(\nu^2 - 1)^{-\frac{1}{2}t}\left(\frac{d}{d\nu}\right)^{-t}\mathrm{P}$ can only differ from P^t by a constant factor. In order to find that factor suppose ν to be infinitely large;

then
$$\mathrm{P} = \frac{2i\,!}{2^i (i\,!)^2}\nu^i,$$
and
$$\mathrm{P}^t = \frac{2i\,!}{2^i\, i\,!} \cdot \frac{\nu^i}{\overline{i - t}\,!}.$$
Also
$$(\nu^2 - 1)^{-\frac{1}{2}t}\left(\frac{d}{d\nu}\right)^{-t}\mathrm{P} = \nu^{-t}\frac{2i\,!}{2^i (i\,!)^2} \cdot \frac{i\,!}{\overline{i + t}\,!}\nu^{i+t} = \frac{2i\,!}{2^i i\,!} \cdot \frac{\nu^i}{\overline{i + t}\,!}.$$

Therefore the factor is $\frac{\overline{i + t}\,!}{\overline{i - t}\,!}$, and
$$\mathrm{P}^t = (\nu^2 - 1)^{\frac{1}{2}t}\left(\frac{d}{d\nu}\right)^t \mathrm{P} = \frac{\overline{i + t}\,!}{\overline{i - t}\,!}(\nu^2 - 1)^{-\frac{1}{2}t}\left(\frac{d}{d\nu}\right)^{-t}\mathrm{P}.$$

It will be convenient to pause here and obtain the corresponding formulæ for the Q-functions. Various writers have adopted various conventions as to the factors involved in these functions. I write
$$\mathrm{Q} = \mathrm{P}\int_\nu^\infty \frac{d\nu}{(\nu^2 - 1)(\mathrm{P})^2},$$
and
$$\mathrm{Q}^t = (\nu^2 - 1)^{\frac{1}{2}t}\left(\frac{d}{d\nu}\right)^t \mathrm{Q}.$$

As in the case of P^t we may change the sign of t, if we introduce a constant factor, and this may be found by making ν infinitely great. In that case it is easy to show that
$$\mathrm{Q} = \frac{2^i (i\,!)^2}{\overline{2i + 1}\,!} \cdot \frac{1}{\nu^{i+1}}.$$

By performing $\left(\frac{d}{d\nu}\right)^t$ and $\left(\frac{d}{d\nu}\right)^{-t}$ on Q it follows that the constant factor is the same as before, and that the alternative forms for Q^t are exactly the same as for P^t.

Hence the transformations which follow for the P-functions are equally applicable to the Q-functions.

If we differentiate P^t in its two forms we find

$$\frac{dP^t}{d\nu} = t\nu(\nu^2-1)^{\frac{1}{2}(t-2)}\left(\frac{d}{d\nu}\right)^t P + (\nu^2-1)^{\frac{1}{2}t}\left(\frac{d}{d\nu}\right)^{t+1} P = \frac{t\nu}{\nu^2-1}P^t + \frac{P^{t+1}}{(\nu^2-1)^{\frac{1}{2}}}.$$

And $$\frac{dP^t}{d\nu} = \frac{i+t\,!}{i-t\,!}\left\{-t\nu(\nu^2-1)^{-\frac{1}{2}(t+2)}\left(\frac{d}{d\nu}\right)^{-t} P + (\nu^2-1)^{-\frac{1}{2}t}\left(\frac{d}{d\nu}\right)^{-(t-1)} P\right\}$$
$$= -\frac{t\nu}{\nu^2-1}P^t + \frac{i+t\,!}{i-t\,!}\cdot\frac{i-t+1\,!}{i+t-1\,!}\,\frac{P^{t-1}}{(\nu^2-1)^{\frac{1}{2}}}.$$

I now write

$$\{i, t\} = (i+t)(i-t+1) = i(i+1) - t(t-1).$$

It is clear that

$$\{i, -t\} = \{i, t+1\}, \text{ and } \{i, 0\} = \{i, 1\} = i(i+1).$$

Now since $\dfrac{i+t\,!}{i-t\,!}\,\dfrac{i-t+1\,!}{i+t-1\,!} = \{i, t\}$, by taking the sum and difference of the two forms of $\dfrac{dP^t}{d\nu}$, we have

$$\left.\begin{aligned}\frac{dP^t}{d\nu} &= \frac{1}{2(\nu^2-1)^{\frac{1}{2}}}\left[P^{t+1} + \{i, t\}P^{t-1}\right],\\ \frac{\nu P^t}{(\nu^2-1)^{\frac{1}{2}}} &= \tfrac{1}{2}\left[-\frac{1}{t}P^{t+1} + \frac{\{i, t\}}{t}P^{t-1}\right]\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (10).$$

It is easy to verify, by means of the relationship $P^{-t} = \dfrac{i-t\,!}{i+t\,!}P^t$, that these equations are true when t is negative. They are also true when $t = 0$, although the second equation then becomes nugatory.

Multiply the first of (10) by ν and the second by $\dfrac{\nu}{(\nu^2-1)^{\frac{1}{2}}}$, and apply them a second time.

Then since
$$\frac{\{i, t+1\}}{t+1} - \frac{\{i, t\}}{t-1} = -\frac{2i(i+1)}{t^2-1},$$
$$\frac{\{i, t+1\}}{t(t+1)} + \frac{\{i, t\}}{t(t-1)} = 2\left(\frac{i(i+1)}{t^2-1} - 1\right),$$
$$\frac{\nu^2+1}{\nu^2-1} = \frac{2\nu^2}{\nu^2-1} - 1\,;$$

$$\left.\begin{aligned}2\nu\frac{dP^t}{d\nu} &= \tfrac{1}{2}\left[-\frac{P^{t+2}}{t+1} - \frac{2i(i+1)}{t^2-1}P^t + \frac{\{i, t\}\{i, t-1\}}{t-1}P^{t-2}\right],\\ \frac{\nu^2+1}{\nu^2-1}P^t &= \tfrac{1}{2}\left[\frac{P^{t+2}}{t(t+1)} - \frac{2i(i+1)}{t^2-1}P^t + \frac{\{i, t\}\{i, t-1\}}{t(t-1)}P^{t-2}\right]\end{aligned}\right\} \quad . \quad . \quad (11).$$

These equations are always true although for $t = \pm 1$ and 0 they become nugatory.

Then

$$-2\nu\frac{d\mathrm{P}^t}{d\nu} + t^2\frac{\nu^2+1}{\nu^2-1}\mathrm{P}^t = \tfrac{1}{2}[\mathrm{P}^{t+2} + \{i,t\}\{i,t-1\}\mathrm{P}^{t-2}] - i(i+1)\mathrm{P}^t.$$

Hence

$$\psi_s(\mathrm{P}^t) = -\tfrac{1}{2}\beta\left[\frac{2(s^2-t^2)}{\beta}\mathrm{P}^t + \mathrm{P}^{t+2} - 2\sigma\mathrm{P}^t + \{i,t\}\{i,t-1\}\mathrm{P}^{t-2}\right]. \qquad (12)$$

(β.) *To find* $\psi_s(\Omega\,\mathrm{P}^t)$.

It is now best to use ψ_s in the form (5), where D_1 is defined by (2).

Now $$\mathrm{D}_1(\Omega\mathrm{P}^t) = \frac{1}{(1-\beta)^{\frac{1}{2}}}\left\{[(\nu^2-1)(1-\beta)-2\beta]\frac{d\mathrm{P}^t}{d\nu} + \frac{2\beta\nu}{\nu^2-1}\mathrm{P}^t\right\},$$

and $$\mathrm{D}_1^{\,2}(\Omega\mathrm{P}^t) = (\nu^2-1)\,\Omega\left\{[(\nu^2-1)(1-\beta)-2\beta]\frac{d^2\mathrm{P}^t}{d\nu^2} + \left(1-\beta+\frac{\beta}{\nu^2-1}\right)2\nu\frac{d\mathrm{P}^t}{d\nu} - \frac{2\beta\mathrm{P}^t}{\nu^2-1}\left(\frac{\nu^2+1}{\nu^2-1}\right)\right\}.$$

The latter terms of ψ_s contribute

$$\Omega\{-i(i+1)[(\nu^2-1)(1-\beta)-\beta]\mathrm{P}^t - (s^2-\beta\sigma)\mathrm{P}^t\}.$$

Therefore

$$\psi_s(\Omega\mathrm{P}^t) = \Omega\left\{(\nu^2-1)[(\nu^2-1)(1-\beta)-2\beta]\frac{d^2\mathrm{P}^t}{d\nu^2} + [(\nu^2-1)(1-\beta)+\beta]\,2\nu\frac{d\mathrm{P}^t}{d\nu} - i(i+1)(1-\beta)(\nu^2-1)\mathrm{P}^t - s^2\mathrm{P}^t + \beta i(i+1)\mathrm{P}^t + \beta\sigma\mathrm{P}^t - 2\beta\mathrm{P}^t\frac{\nu^2+1}{\nu^2-1}\right\}.$$

But $(\nu^2-1)\dfrac{d^2\mathrm{P}^t}{d\nu^2} = -2\nu\dfrac{d\mathrm{P}^t}{d\nu} + i(i+1)\mathrm{P}^t + \dfrac{t^2}{\nu^2-1}\mathrm{P}^t$, and we find on reduction that

$$\psi_s(\Omega\mathrm{P}^t) = \Omega\mathrm{P}^t(t^2-s^2) - \beta\Omega\left[-6\nu\frac{d\mathrm{P}^t}{d\nu} + i(i+1)\mathrm{P}^t + \frac{\nu^2+1}{\nu^2-1}(t^2+2)\mathrm{P}^t - \sigma\mathrm{P}^t\right].$$

On substituting for $\dfrac{\nu d\mathrm{P}^t}{d\nu}$ and $\dfrac{\nu^2+1}{\nu^2-1}\mathrm{P}^t$ their values, we have

$$\psi_s(\Omega\mathrm{P}^t) = -\tfrac{1}{2}\beta\Omega\left[\frac{2(s^2-t^2)}{\beta}\mathrm{P}^t + \frac{t+2}{t}\mathrm{P}^{t+2} - 2\sigma\mathrm{P}^t + \frac{t-2}{t}\{i,t\}\{i,t-1\}\mathrm{P}^{t-2}\right]. \quad . \quad . \quad . \quad (13).$$

(γ.) *To find* $\chi_s\left(\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} t\phi\right)$.

In this case the most convenient form for χ_s is that in (9), and we easily find

$$\chi_s\left(\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} t\phi\right) = -\tfrac{1}{2}\beta\left\{-\frac{2(s^2-t^2)}{\beta}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} t\phi + \{i, t+1\}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(t+2)\phi\right.$$
$$\left. + 2\sigma\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} t\phi + \{i, t\}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(t-2)\phi\right\} \quad . \quad (14).$$

(δ.) *To find* $\chi_s\left(\Phi\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} t\phi\right)$.

I now use the form χ_s as defined in (5), where D_3 is given in (2), so that $D_3 = (1-\beta\cos 2\phi)^{\frac{1}{2}}\dfrac{d}{d\phi} = \Phi\dfrac{d}{d\phi}$.

We have

$$D_3\left(\Phi\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} t\phi\right) = \mp t\begin{Bmatrix}\sin\\ \cos\end{Bmatrix} t\phi \pm \tfrac{1}{2}\beta(t+1)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(t+2)\phi \pm \tfrac{1}{2}\beta(t-1)\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(t-2)\phi$$

and

$$D_3^2\left(\Phi\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} t\phi\right) = \Phi\left[-t^2\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} t\phi + \tfrac{1}{2}\beta(t+1)(t+2)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(t+2)\phi\right.$$
$$\left. + \tfrac{1}{2}\beta(t-1)(t-2)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(t-2)\phi\right].$$

The latter terms of χ_s contribute

$$\Phi\left\{(s^2-\beta\sigma)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} t\phi - \tfrac{1}{2}\beta i(i+1)\left[\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(t+2)\phi + \begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(t-2)\phi\right]\right\}$$

Therefore

$$\chi_s\left(\Phi\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} t\phi\right) = -\tfrac{1}{2}\beta\Phi\left\{-\frac{2(s^2-t^2)}{\beta}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} t\phi + \{i, t+2\}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(t+2)\phi\right.$$
$$\left. + 2\sigma\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} t\phi + \{i, t-1\}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(t-2)\phi\right\} \quad . \quad (15).$$

§ 6. *Determination of the Coefficients in the Functions.*

In this section I use successively the four results (12) (13) (14) (15) obtained in the last section under the headings (α), (β), (γ), (δ).

(α) $$\mathfrak{P}^s = q_s P^s + \Sigma\beta^n q_{s-2n} P^{s-2n} + \Sigma\beta^n q_{s+2n} P^{s+2n}.$$

The limits of the first Σ are 1 to $\frac{1}{2}s$ or $\frac{1}{2}(s-1)$, and of the second 1 to $\frac{1}{2}(i-s)$ or $\frac{1}{2}(i-s-1)$.

Applying the operation ψ_s to $\mathfrak{P}^s$ and equating $-\dfrac{2}{\beta}\psi_s(\mathfrak{P}^s)$ to zero, we have

$$\Sigma 8n(s-n)\beta^{n-1}q_{s-2n}P^{s-2n} - \Sigma 8n(s+n)\beta^{n-1}q_{s+2n}P^{s+2n}$$
$$+ q_s[P^{s+2} - 2\sigma P^s + \{i, s\}\{i, s-1\}P^{s-2}]$$
$$+ \Sigma\beta^n q_{s-2n}[P^{s-2n+2} - 2\sigma P^{s-2n} + \{i, s-2n\}\{i, s-2n-1\}P^{s-2n-2}]$$
$$+ \Sigma\beta^n q_{s+2n}[P^{s+2n+2} - 2\sigma P^{s+2n} + \{i, s+2n\}\{i, s+2n-1\}P^{s+2n-2}] = 0.$$

The coefficients of the P's must vanish separately. This gives from the coefficients of P^{s-2n} and P^{s+2n} the following :—

$$2[4n(s-n) - \beta\sigma]q_{s-2n} + \beta^2 q_{s-2n-2} + \{i, s-2n+2\}\{i, s-2n+1\}q_{s-2n+2} = 0,$$

$$-2[4n(s+n) + \beta\sigma]q_{s+2n} + q_{s+2n-2} + \beta^2\{i, s+2n+2\}\{i, s+2n+1\}q_{s+2n+2} = 0.$$

These equations may be written in the form

$$\left.\begin{aligned} \frac{2q_{s-2n}}{q_{s-2n+2}} &= \frac{-\{i, s-2n+2\}\{i, s-2n+1\}}{4n(s-n) - \beta\sigma + \frac{1}{4}\beta^2\left(\frac{2q_{s-2n-2}}{q_{s-2n}}\right)}, \\ \frac{2q_{s+2n}}{q_{s+2n-2}} &= \frac{1}{4n(s+n) + \beta\sigma - \frac{1}{4}\beta^2\{i, s+2n+2\}\{i, s+2n+1\}\left(\frac{2q_{s+2n+2}}{q_{s+2n}}\right)} \end{aligned}\right\} \quad . \quad (16).$$

Whence by continued application, the continued fractions

$$\left.\begin{aligned} \frac{2q_{s-2n}}{q_{-2n+2}} &= \frac{-\{i, s-2n+2\}\{i, s-2n+1\}}{4n(s-n) - \beta\sigma -}\ \frac{\frac{1}{4}\beta^2\{i, s-2n\}\{i, s-2n-1\}}{4(n+1)(s-n-1) - \ldots} \\ &\qquad \ldots \frac{-\frac{1}{4}\beta^2\{i, s-2n-2r+2\}\{i, s-2n-2r+1\}}{4(n+r)(s-n-r) + \frac{1}{4}\beta^2\left(\frac{2q_{s-2n-2r-2}}{q_{s-2n-2r}}\right)}, \\ \frac{2q_{s+2n}}{q_{s+2n-2}} &= \frac{1}{4n(s+n) + \beta\sigma -}\ \frac{\frac{1}{4}\beta^2\{i, s+2n+2\}\{i, s+2n+1\}}{4(n+1)(s+n+1) + \beta\sigma - \ldots} \\ &\qquad \ldots \frac{-\frac{1}{4}\beta^2\{i, s+2n+2r\}\{i, s+2n+2r-1\}}{4(n+r)(s+n+r) - \frac{1}{4}\beta^2\{i, s+2n+2r+2\}\{i, s+2n+2r+1\}\left(\frac{2q_{s+2n+2r+2}}{q_{s+2n+2r}}\right)} \end{aligned}\right\} (16).$$

We must now consider what I may call the middle of the series, which corresponds with $n = 0$. In this case each of the Σ's contributes one term and the q_s term gives another. The result is

$$-2\sigma q_s + \beta q_{s-2} + \beta q_{s+2}\{i, s+2\}\{i, s+1\} = 0,$$

or
$$\beta\sigma = \tfrac{1}{4}\beta^2\left(\frac{2q_{s-2}}{q_s}\right) + \tfrac{1}{4}\beta^2\{i, s+2\}\{i, s+1\}\left(\frac{2q_{s+2}}{q_s}\right).$$

Since $2q_{s-2}/q_s$ and $2q_{s+2}/q_s$ are expressible as continued fractions, we have an equation for $\beta\sigma$, if the continued fractions terminate.

We shall now consider those terminations.

First, suppose that s is even, corresponding to types EEC, OEC.

The first continued fraction depends only on the first Σ. The condition to be satisfied is

$$\begin{aligned}&2s^2\beta^{\frac{1}{2}(s-2)}q_0\mathrm{P} + 2(s^2-4)\beta^{\frac{1}{2}(s-4)}q_2\mathrm{P}^2 + \ldots \\ &\quad + \beta^{\frac{1}{2}s}q_0[\mathrm{P}^2 - 2\sigma\mathrm{P} + \{i, 0\}\{i, -1\}\mathrm{P}^{-2}] \\ &\quad + \beta^{\frac{1}{2}(s-2)}q_2[\mathrm{P}^4 - 2\sigma\mathrm{P}^2 + \{i, 2\}\{i, 1\}\mathrm{P}] \\ &\quad + \beta^{\frac{1}{2}(s-4)}q_4[\mathrm{P}^6 - 2\sigma\mathrm{P}^4 + \{i, 4\}\{i, 3\}\mathrm{P}^2] + \ldots = 0.\end{aligned}$$

Since $\{i, 0\}\{i, -1\}\mathrm{P}^{-2} = \mathrm{P}^2$, we have, by equating to zero the coefficients of P and P^2, results which may be written

$$\frac{2q_0}{q_2} = \frac{-\{i, 2\}\{i, 1\}}{s^2 - \beta\sigma}, \qquad \frac{2q_2}{q_4} = \frac{-\{i, 4\}\{i, 3\}}{s^2 - 4 - \beta\sigma + \frac{1}{2}\beta^2\left(\frac{2q_0}{q_2}\right)}.$$

Hence the q's disappear from the first continued fraction, which terminates with

$$\frac{-\frac{1}{2}\beta^2\{i, 2\}\{i. 1\}}{s^2 - \beta\sigma}.$$

In this last term the $\frac{1}{4}\beta^2$ which prevails elsewhere is replaced by $\frac{1}{2}\beta^2$.

Observe that when $s = 2$ the first continued fraction is replaced by a simple fraction, so that the equation for $\beta\sigma$ becomes

$$\beta\sigma = \frac{-\frac{1}{2}\beta^2\{i, 2\}\{i, 1\}}{4 - \beta\sigma} + \tfrac{1}{4}\beta^2\{i, 4\}\{i, 3\}\left(\frac{2q_4}{q_2}\right).$$

Secondly, suppose that s is odd, corresponding to the types OOS, EOS.

The condition to be satisfied is now

$$\begin{aligned}&2(s^2-1)\beta^{\frac{1}{2}(s-3)}q_1\mathrm{P}^1 + 2(s^2-9)\beta^{\frac{1}{2}(s-5)}q_3\mathrm{P}^3 + \ldots \\ &\quad + \beta^{\frac{1}{2}(s-1)}q_1[\mathrm{P}^3 - 2\sigma\mathrm{P}^1 + \{i, 1\}\{i, 0\}\mathrm{P}^{-1}] \\ &\quad + \beta^{\frac{1}{2}(s-3)}q_3[\mathrm{P}^5 - 2\sigma\mathrm{P}^3 + \{i, 3\}\{i, 2\}\mathrm{P}^1] \\ &\quad + \beta^{\frac{1}{2}(s-5)}q_5[\mathrm{P}^7 - 2\sigma\mathrm{P}^5 + \{i, 5\}\{i, 4\}\mathrm{P}^3] + \ldots = 0.\end{aligned}$$

Now $\{i, 1\}\{i, 0\}\mathrm{P}^{-1} = i(i+1)\frac{i+1\,!}{i-1\,!}\mathrm{P}^{-1} = i(i+1)\mathrm{P}^1$, and if we equate to zero the coefficients of P^1 and P^3 we obtain results which may be written

$$\frac{2q_1}{q_3} = \frac{-\{i, 3\}\{i, 2\}}{s^2 - 1 - \beta\sigma + \frac{1}{2}\beta i(i+1)},$$

$$\frac{2q_3}{q_5} = \frac{-\{i, 5\}\{i, 4\}}{s^2 - 9 - \beta\sigma + \frac{1}{4}\beta^2\left(\frac{2q_1}{q_3}\right)}.$$

Thus the q's again disappear, and the first continued fraction ends with

$$\frac{-\frac{1}{4}\beta^2\{i, 3\}\{i, 2\}}{s^2 - 1 - \beta\sigma + \frac{1}{2}\beta i(i+1)}.$$

Observe that when $s = 3$, the continued fraction reduces to a simple fraction, and the equation for $\beta\sigma$ becomes

$$\beta\sigma = \frac{-\frac{1}{4}\beta^2\{i, 3\}\{i, 2\}}{8 - \beta\sigma + \frac{1}{2}\beta i(i+1)} + \frac{1}{4}\beta^2\{i, 5\}\{i, 4\}\left(\frac{2q_5}{q_3}\right).$$

The case of $s = 1$ must be considered separately.

We have next to consider the termination of the second fraction, which depends only on the second Σ.

First, when i and s are either both even or both odd, the types are EEC and OOS, and the limits are $\frac{1}{2}(i-s)$ to 1. The condition to be satisfied is

$$\begin{aligned} &-2(i^2 - s^2)\beta^{\frac{1}{2}(i-s-2)}q_i\mathrm{P}^i - 2[(i-2)^2 - s^2]\beta^{\frac{1}{2}(i-s-4)}q_{i-2}\mathrm{P}^{i-2} - \dots \\ &+ \beta^{\frac{1}{2}(i-s)}q_i[\mathrm{P}^{i+2} - 2\sigma\mathrm{P}^i + \{i, i\}\{i, i-1\}\mathrm{P}^{i-2}] \\ &+ \beta^{\frac{1}{2}(i-s-2)}q_{i-2}[\mathrm{P}^i - 2\sigma\mathrm{P}^{i-2} + \{i, i-2\}\{i, i-3\}\mathrm{P}^{i-4}] \\ &+ \beta^{\frac{1}{2}(i-s-4)}q_{i-4}[\mathrm{P}^{i-2} - 2\sigma\mathrm{P}^{i-4} + \{i, i-4\}\{i, i-5\}\mathrm{P}^{i-6}] + \dots = 0. \end{aligned}$$

Now P^{i+2} is zero, and equating the coefficients of P^i and P^{i-2} to zero we obtain results which may be written

$$\frac{2q_i}{q_{i-2}} = \frac{1}{i^2 - s^2 + \beta\sigma},$$

$$\frac{2q_{i-2}}{q_{i-4}} = \frac{1}{(i-2)^2 - s^2 + \beta\sigma - \frac{1}{4}\beta^2\{i, i\}\{i, i-1\}\left(\frac{2q_i}{q_{i-2}}\right)}.$$

Hence this continued fraction ends with

$$\frac{-\frac{1}{4}\beta^2\{i, i\}\{i, i-1\}}{i^2 - s^2 + \beta\sigma}.$$

Secondly, when i and s differ as to evenness or oddness, the types are OEC and EOS, and the limits are $\frac{1}{2}(i-s-1)$ to 1. The same investigation applies again when i is changed into $i - 1$.

Hence the continued fraction ends with

$$\frac{-\frac{1}{4}\beta^2\{i, i-1\}\{i, i-2\}}{(i-1)^2 - s^2 + \beta\sigma}.$$

The cases of $s = 0$, $s = 1$ must be considered by themselves.

When $s = 0$, the types are EEC and OEC. The "middle" of the series is now also an end, and the condition is

$$- 8q_2 P^2 - 8 . 2^2 \beta q_4 P^4 - \ldots + q_0 [P^2 - 2\sigma P + \{i, 0\}\{i, -1\} P^{-2}]$$
$$+ \beta q_2 [P^4 - 2\sigma P^2 + \{i, 2\}\{i, 1\} P] + \beta^2 q_4 [P^6 - 2\sigma P^4 + \{i, 4\}\{i, 3\} P^2] + \ldots = 0.$$

Writing P^2 for $\{i, 0\}\ \{i, -1\}\ P^{-2}$ and equating the coefficients of P and P^2 to zero, we have

$$\beta\sigma = \tfrac{1}{2}\beta^2 \{i, 1\}\{i, 2\} \left(\frac{q_2}{q_0}\right),$$

$$\frac{q_2}{q_0} = \frac{1}{4 + \beta\sigma - \frac{1}{4}\beta^2 \{i, 3\}\{i, 4\} \left(\frac{2q_4}{q_2}\right)}.$$

Therefore $$\beta\sigma = \frac{\frac{1}{2}\beta^2 \{i, 1\}\{i, 2\}}{4 . 1^2 + \beta\sigma -}\ \frac{\frac{1}{4}\beta^2 \{i, 3\}\{i, 4\}}{4 . 2^2 + \beta\sigma -}\ \frac{-\frac{1}{4}\beta^2 \{i, 5\}\{i, 6\}}{4 . 3^2 + \beta\sigma - \ldots},$$

ending with $\frac{-\frac{1}{4}\beta^2 \{i, i\}\{i, i-1\}}{i^2 + \beta\sigma}$ for EEC, and with $\frac{-\frac{1}{4}\beta^2 \{i, i-1\}\{i, i-2\}}{(i-1)^2 + \beta\sigma}$ for OEC.

Next when $s = 1$ the types are OOS, EOS; the "middle" is again an end, and the condition is

$$- 8 . 1 . 2 q_3 P^3 - 8 . 2 . 3 \beta q_5 P^5 - \ldots + q_1 [P^3 - 2\sigma P^1 + \{i, 1\}\{i, 0\} P^{-1}]$$
$$+ \beta q_3 [P^5 - 2\sigma P^3 + \{i, 3\}\{i, 2\} P^1] + \beta^2 q_5 [P^7 - 2\sigma P^5 + \{i, 5\}\{i, 4\} P^3] + \ldots = 0.$$

Writing $i(i + 1)\ P^1$ for $\{i, 1\}\ \{i, 0\}\ P^{-1}$ and equating to zero the coefficients of P^1 and P^3, we have

$$\beta\sigma - \tfrac{1}{2}\beta i (i + 1) = \tfrac{1}{4}\beta^2 \{i, 3\}\{i, 2\} \left(\frac{2q_3}{q_1}\right),$$

$$\frac{2q_3}{q_1} = \frac{1}{4 . 1 . 2 + \beta\sigma - \frac{1}{4}\beta^2 \{i, 5\}\{i, 4\} \left(\frac{2q_5}{q_3}\right)}.$$

Therefore

$$\beta\sigma - \tfrac{1}{2}\beta i (i + 1) = \frac{\frac{1}{4}\beta^2 \{i, 3\}\{i, 2\}}{4 . 1 . 2 + \beta\sigma -}\ \frac{\frac{1}{4}\beta^2 \{i, 5\}\{i, 4\}}{4 . 2 . 3 + \beta\sigma -}\ \frac{\frac{1}{4}\beta^2 \{i, 7\}\{i, 6\}}{4 . 3 . 4 + \beta\sigma - \ldots}$$

ending with $\frac{-\frac{1}{4}\beta^2 \{i, i\}\{i, i-1\}}{i^2 + \beta\sigma}$ for OOS, and with $\frac{-\frac{1}{4}\beta^2 \{i, i-1\}\{i, i-2\}}{(i-1)^2 + \beta\sigma}$ for EOS.

(β.) We have next to consider the other form of P-function for types EES, OES, OOC, EOC, namely,

$$\mathsf{P}^s = \Omega [q'_s P^s + \Sigma \beta^n q'_{s-2n} P^{s-2n} + \Sigma \beta^n q'_{s+2n} P^{s+2n}],$$

where $$\Omega = \left(\frac{\nu^2 - \frac{1+\beta}{1-\beta}}{\nu^2 - 1}\right)^{\frac{1}{2}}.$$

Let us write $q'_{s\pm 2n} = (s \pm 2n)\, q_{s\pm 2n}$. The q's are not now the actual coefficients of any P-function, but we shall see that they are determinable by almost the same relationships as those already found, and therefore the notation is convenient.

We now have

$$\mathsf{P}^s = \Omega\,[q_s s \mathrm{P}^s + \Sigma\beta^n q_{s-2n}(s-2n)\,\mathrm{P}^{s-2n} + \Sigma\beta^n q_{s+2n}(s+2n)\,\mathrm{P}^{s+2n}].$$

Applying the operation ψ_s to P^s and equating $-\dfrac{2}{\Omega\beta}\psi_s(\mathsf{P}^s)$ to zero, we have

$$\begin{aligned}
&\Sigma 8n(s-n)(s-2n)\,q_{s-2n}\mathrm{P}^{s-2n}\Sigma 8n(s+n)(s+2n)\,q_{s+2n}\mathrm{P}^{s+2n} \\
&+ q_s[(s+2)\,\mathrm{P}^{s+2} - 2\sigma s\mathrm{P}^s + \{i, s\}\{i, s-1\}(s-2)\,\mathrm{P}^{s-2}] \\
&+ \Sigma\beta^n q_{s-2n}[(s-2n+2)\,\mathrm{P}^{s-2n+2} - 2\sigma(s-2n)\,\mathrm{P}^{s-2n} \\
&\qquad\qquad + \{i, s-2n\}\{i, s-2n-1\}(s-2n-2)\,\mathrm{P}^{s-2n-2}] \\
&+ \Sigma\beta^n q_{s+2n}[(s+2n+2)\,\mathrm{P}^{s+2n+2} - 2\sigma(s+2n)\,\mathrm{P}^{s+2n} \\
&\qquad\qquad + \{i, s+2n\}\{i, s+2n-1\}(s+2n-2)\,\mathrm{P}^{s+2n-2}] = 0.
\end{aligned}$$

This is the same equation as before, if we replace $t\mathrm{P}^t$ by P^t. As we may equate coefficients of $t\mathrm{P}^t$ to zero (instead of coefficients of P^t), we obtain the same equations for the q's as before.

A certain change must, however, be noted with respect to the beginning of the first series, which determines the end of the first continued fraction.

We previously wrote P^2 for $\{i, 0\}\{i, -1\}\,\mathrm{P}^{-2}$ and $i(i+1)\,\mathrm{P}^1$ for $\{i, 1\}\{i, 0\}\mathrm{P}^{-1}$. But the corresponding terms will now be $\{i, 0\}\{i, -1\}(-2)\,\mathrm{P}^{-2}$ and $\{i, 1\}\{i, 0\}(-1)\,\mathrm{P}^{-1}$, and these are equal to $-(2\mathrm{P}^2)$ and $-(1\,.\,\mathrm{P}^1)$.

Hence it follows that when s is even (EES, OES)

$$\frac{2q_2}{q_0} = \frac{-\{i, 2\}\{i, 1\}}{s^2-\beta\sigma}, \qquad \frac{2q_2}{q_4} = \frac{-\{i, 4\}\{i, 3\}}{s^2-4-\beta\sigma}.$$

The q_0 term has disappeared from the latter of these, and thus the continued fraction is independent of q_0. This is correct, since whatever value (short of infinity) q_0 may have q'_0, being equal to $0q_0$, vanishes. Hence the continued fraction is docked of one term and ends with

$$\frac{-\tfrac{1}{4}\beta^2\{i, 4\}\{i, 3\}}{s^2-4-\beta\sigma}.$$

It is important to note the deficiency of one term in the fraction, since it indicates that when $s = 2$ the first continued fraction entirely disappears.

When $s = 0$ there is no function of the P form, so the question of interpretation does not arise.

When s is odd (OOC, EOC) the only change is that $i(i+1)$ enters with the opposite sign, so that the first fraction ends with

$$\frac{-\frac{1}{4}\beta^2\{i, 3\}\{i, 2\}}{s^2 - 1 - \beta\sigma - \frac{1}{2}\beta i(i+1)}.$$

When $s = 1$, we have $\beta\sigma + \frac{1}{2}\beta i(i+1)$ equal to the same fraction as before.

When the q's are determined we have $q'_t = tq_t$. But it is desired that in the case (α) q_s should be unity, and that in the case (β) q'_s should be unity. This condition will be satisfied in the present case if we determine all the q's, put q_s equal to unity, and finally take

$$q'_{s \pm 2n} = \frac{s \pm 2n}{s} q_{s \pm 2n}.$$

Thus in both (α) and (β) we put q_s equal to unity, and in (β) determine the q''s by the above equation.

(γ.) We now have to consider the cosine and sine functions.

For EEC, EES, OOC, OOS

$$\left\{\begin{matrix}\mathfrak{C}^s \\ \mathfrak{S}^s\end{matrix}\right. = p_s \left\{\begin{matrix}\cos \\ \sin\end{matrix}\right. s\phi + \Sigma\beta^n p_{s-2n} \left\{\begin{matrix}\cos \\ \sin\end{matrix}\right. (s-2n)\phi + \Sigma\beta^n p_{s+2n} \left\{\begin{matrix}\cos \\ \sin\end{matrix}\right. (s+2n)\phi.$$

The first Σ has limits $\frac{1}{2}s$ or $\frac{1}{2}(s-1)$ to 1, the second $\frac{1}{2}(i-s)$ to 1.

Apply the operation χ_s and equate $-\dfrac{2}{\beta}\chi_s\left(\left\{\begin{matrix}\mathfrak{C}^s \\ \mathfrak{S}^s\end{matrix}\right.\right)$ to zero; then

$$\begin{aligned}
&- \Sigma 8n(s-n)\beta^{n-1}p_{s-2n}\left\{\begin{matrix}\cos \\ \sin\end{matrix}\right.(s-2n)\phi + \Sigma 8n(s+n)\beta^{n-1}p_{s+2n}\left\{\begin{matrix}\cos \\ \sin\end{matrix}\right.(s+2n)\phi \\
&+ p_s\left[\{i, s+1\}\left\{\begin{matrix}\cos \\ \sin\end{matrix}\right.(s+2)\phi + 2\sigma\left\{\begin{matrix}\cos \\ \sin\end{matrix}\right. s\phi + \{i, s\}\left\{\begin{matrix}\cos \\ \sin\end{matrix}\right.(s-2)\phi\right] \\
&+ \Sigma\beta^n p_{-2n}\left[\{i, s-2n+1\}\left\{\begin{matrix}\cos \\ \sin\end{matrix}\right.(s-2n+2)\phi + 2\sigma\left\{\begin{matrix}\cos \\ \sin\end{matrix}\right.(s-2n)\phi\right. \\
&\qquad\qquad\qquad\left. + \{i, s-2n\}\left\{\begin{matrix}\cos \\ \sin\end{matrix}\right.(s-2n-2)\phi\right] \\
&+ \Sigma\beta^n p_{s+2n}\left[\{i, s+2n+1\}\left\{\begin{matrix}\cos \\ \sin\end{matrix}\right.(s+2n+2)\phi + 2\sigma\left\{\begin{matrix}\cos \\ \sin\end{matrix}\right.(s+2n)\phi\right. \\
&\qquad\qquad\qquad\left. + \{i, s+2n\}\left\{\begin{matrix}\cos \\ \sin\end{matrix}\right.(s+2n-2)\phi\right] = 0.
\end{aligned}$$

If we equate to zero the coefficients of $\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(s \pm 2n)\phi$, we find

$$\frac{2p_{s-2n}}{p_{s-2n+2}} = \frac{\{i, s-2n+2\}}{4n(s-n) - \beta\sigma - \frac{1}{4}\beta^2\{i, s-2n-1\}\left(\dfrac{2p_{s-2n-2}}{p_{s-2n}}\right)},$$

$$\frac{2p_{s+2n}}{p_{s+2n-2}} = \frac{-\{i, s+2n-1\}}{4n(s+n) + \beta\sigma + \frac{1}{4}\beta^2\{i, s+2n+2\}\left(\dfrac{2p_{s+2n+2}}{p_{s+2n}}\right)}.$$

These will, as before, lead to continued fractions, and by elimination of the p's to an equation for $\beta\sigma$. The equation will agree with our former result, for it can of course make no difference from which equation we determine σ.* It follows then by comparison with the previous result (16) that

$$\frac{p_{s-2n}}{p_{s-2n+2}} = -\frac{1}{\{i, s-2n+1\}}\frac{q_{s-2n}}{q_{s-2n+2}},$$

$$\frac{p_{s+2n}}{p_{s+2n-2}} = -\{i, s+2n-1\}\frac{q_{s+2n}}{q_{s+2n-2}}.$$

Hence when the q's are found, the p's follow at once.

(δ.) For OEC, OES, EOC, EOS

$$\begin{Bmatrix}\mathbf{C}^s\\ \mathbf{S}^s\end{Bmatrix} = \Phi\left[p_s'\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}s\phi + \Sigma\beta^n p'_{s-2n}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(s-2n)\phi + \Sigma\beta^n p'_{s+2n}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(s+2n)\phi\right],$$

where $\Phi = (1 - \beta\cos 2\phi)^{\frac{1}{2}}$.

The limits of the first Σ are $\frac{1}{2}s$ or $\frac{1}{2}(s-1)$ to 1, of the second $\frac{1}{2}(i-s-1)$ to 1. Proceeding exactly as before we find

$$\frac{2p'_{s-2n}}{p'_{s-2n+2}} = \frac{\{i, s-2n+1\}}{4n(s-n) - \beta\sigma - \frac{1}{4}\beta^2\{i, s-2n\}\left(\dfrac{2p'_{s-2n-2}}{p'_{s-2n}}\right)},$$

$$\frac{2p'_{s+2n}}{p'_{s+2n-2}} = \frac{-\{i, s+2n\}}{4n(s+n) + \beta\sigma + \frac{1}{4}\beta^2\{i, s+2n+1\}\left(\dfrac{2p'_{s+2n+}}{p'_{s+2n}}\right)}.$$

By comparison with (16) we see that

$$\frac{p'_{s-2n}}{p'_{s-2n+2}} = -\frac{1}{\{i, s-2n+2\}}\frac{q_{s-2n}}{q_{s-2n+2}},$$

$$\frac{p'_{s+2n}}{p'_{s+2n-2}} = -\{i, s+2n\}\frac{q_{s+2n}}{q_{s+2n-2}}.$$

Therefore when the q's are found, the p's follow at once.

* I have of course verified that this is so.

We may now summarise our results, as follows :—

In the general case where s is neither 0 nor 1, $\beta\sigma$ is the root which nearly vanishes of the equation

$$\beta\sigma = \frac{-\frac{1}{4}\beta^2\{i, s\}\{i, s-1\}}{4.1(s-1)-\beta\sigma-}\;\frac{\frac{1}{4}\beta^2\{i, s-2\}\{i, s-3\}}{4.2(s-2)-\beta\sigma-\ldots}$$
$$+\frac{\frac{1}{4}\beta^2\{i, s+1\}\{i, s+2\}}{4.1(s+1)+\beta\sigma-}\;\frac{\frac{1}{4}\beta^2\{i, s+3\}\{i, s+4\}}{4.2(s+2)+\beta\sigma-\ldots}.$$

The continued fractions terminate variously for the various types of function. The end of the first continued fraction is as follows :—

For EEC $\dfrac{-\frac{1}{2}\beta^2\{i, 1\}\{i, 2\}}{s^2-\beta\sigma}$; and when $s = 2$ this is the whole fraction.

For EES $\dfrac{-\frac{1}{4}\beta^2\{i, 3\}\{i, 4\}}{s^2-4-\beta\sigma}$; and when $s = 2$ the fraction disappears.

For OOC $\dfrac{-\frac{1}{4}\beta^2\{i, 2\}\{i, 3\}}{s^2-1-\beta\sigma-\frac{1}{2}\beta i(i+1)}$; and when $s = 3$ this is the whole fraction.

For OOS $\dfrac{-\frac{1}{4}\beta^2\{i, 2\}\{i, 3\}}{s^2-1-\beta\sigma+\frac{1}{2}\beta i(i+1)}$; and when $s = 3$ this is the whole fraction.

For OEC $\dfrac{-\frac{1}{2}\beta^2\{i, 1\}\{i, 2\}}{s^2-\beta\sigma}$; and when $s = 2$ this is the whole fraction.

For OES $\dfrac{-\frac{1}{4}\beta^2\{i, 3\}\{i, 4\}}{s^2-4-\beta\sigma}$; and when $s = 2$ the fraction disappears.

For EOC $\dfrac{-\frac{1}{4}\beta^2\{i, 2\}\{i, 3\}}{s^2-1-\beta\sigma-\frac{1}{2}\beta i(i+1)}$; and when $s = 3$ this is the whole fraction.

For EOS $\dfrac{-\frac{1}{4}\beta^2\{i, 2\}\{i, 3\}}{s^2-1-\beta\sigma+\frac{1}{2}\beta i(i+1)}$; and when $s = 3$ this is the whole fraction.

For the first four of these types, viz., EEC, EES, OOC, OOS, the second continued fraction ends with

$\dfrac{-\frac{1}{4}\beta^2\{i, i\}\{i, i-1\}}{i^2-s^2+\beta\sigma}$; and when $s = i$ this is the whole fraction, but with the sign changed.

For the last four, viz., OEC, OES, EOC, EOS, it ends with

$\dfrac{-\frac{1}{4}\beta^2\{i, i-1\}\{i, i-2\}}{(i-1)^2-s^2+\beta\sigma}$; and when $s = i-1$ this is the whole fraction, but with the sign changed.

When $s = 0$, the equation becomes

$$\beta\sigma = \frac{\frac{1}{2}\beta^2 \{i, 1\} \{i, 2\}}{4.1^2 + \beta\sigma -} \quad \frac{\frac{1}{4}\beta^2 \{i, 3\} \{i, 4\}}{4.2^2 + \beta\sigma - \ldots},$$

ending when i is even (EEC) with $\dfrac{-\frac{1}{4}\beta^2 \{i, i\} \{i, i-1\}}{i^2 + \beta\sigma}$;

and when i is odd (OEC) with $\dfrac{-\frac{1}{4}\beta^2 \{i, i-1\} \{i, i-2\}}{(i-1)^2 + \beta\sigma}$.

When $s = 1$ the equation has two forms, which may, however, be written together. If the upper sign refers to cosines (OOC, EOC) and the lower to sines (OOS, EOS), the equations are:—

$$\beta\sigma \pm \tfrac{1}{2}\beta i (i + 1) = \frac{\frac{1}{4}\beta^2 \{i, 2\} \{i, 3\}}{4.1.2 + \beta\sigma -} \quad \frac{\frac{1}{4}\beta^2 \{i, 4\} \{i, 5\}}{4.2.3 + \beta\sigma - \ldots},$$

ending when i is even (EOC, EOS) with $\dfrac{-\frac{1}{4}\beta^2 \{i, i-1\} \{i, i-2\}}{(i-1)^2 - 1 + \beta\sigma}$;

and when i odd (OOC, OOS) with $\dfrac{-\frac{1}{4}\beta^2 \{i, i\} \{i, i-1\}}{i^2 - 1 + \beta\sigma}$.

It might appear at first sight that a difficulty will arise in the interpretation of these results when i is small, for the numbers in the denominators of the fractions increase, and yet it is possible that the number at the end should be smaller than that at the beginning; thus apparently the fraction ends before it begins. But this difficulty does not really arise, because in such cases the numerator will always be found to vanish, and thus the whole fraction disappears. For example, in the last case specified, if $s = 1$, $i = 2$ the denominators, according to the formula, begin with $8 + \beta\sigma$ and end with $0 + \beta\sigma$; but the fraction has for numerator $\{2, 2\} \{2, 3\}$, which vanishes.

When $\beta\sigma$ has been determined we find the q's by the formulæ—

$$\frac{2q_{s-2n}}{q_{s-2n+2}} = \frac{-\{i, s-2n+2\} \{i, s-2n+1\}}{4n(s-n) - \beta\sigma -} \quad \frac{\frac{1}{4}\beta^2\{i, s-2n\} \{i, s-2n-1\}}{4(n+1)(s-n-1) - \beta\sigma - \ldots},$$

$$\frac{2q_{s+2n}}{q_{s+2n-2}} = \frac{1}{4n(s+n) + \beta\sigma -} \quad \frac{\frac{1}{4}\beta^2\{i, s+2n+2\} \{i, s+2n+1\}}{4(n+1)(s+n+1) + \beta\sigma - \ldots}.$$

The terminations of the continued fractions are as specified above in the equation for $\beta\sigma$.

By forming continued products of ratios of successive q's, we can find all the q's as multiples of q_s, and $q_s = 1$.

In the cases EEC, OEC, OOS, EOS, these are the required coefficients for $\mathfrak{P}^s$.

In the cases EES, OES, OOC, EOS we put $q_{s\ \pm 2n}' = \frac{s \pm 2n}{s} q_{s \pm 2n}$, and thus find the coefficients for P^s.

The coefficients for $\mathfrak{C}$, $\mathfrak{S}$ in EEC, EES, OOC, OOS are determined by

$$\frac{p_{s-2n}}{p_{s-2n+2}} = -\frac{1}{\{i, s-2n+1\}} \frac{q_{s-2n}}{q_{s-2n+2}},$$

$$\frac{p_{s+2n}}{p_{s+2n-2}} = -\{i, s+2n-1\} \frac{q_{s+2n}}{q_{s+2n-2}}.$$

The coefficients for C, S in OEC, OES, EOC, EOS are determined by

$$\frac{p'_{s-2n}}{p'_{s-2n+2}} = -\frac{1}{\{i, s-2n+2\}} \frac{q_{s-2n}}{q_{s-2n+2}},$$

$$\frac{p'_{s+2n}}{p'_{s+2n-2}} = -\{i, s+2n\} \frac{q_{s+2n}}{q_{s+2n-2}}.$$

It follows that if we put $q_s = 1$ and $p_s = 1$

$$p_{s-2n} = (-)^n \frac{1}{\{i, s-2n+1\}\{i, s-2n+3\} \ldots \{i, s-1\}} q_{s-2n},$$

$$p_{s+2n} = (-)^n \{i, s+2n-1\}\{i, s+2n-3\} \ldots \{i, s+1\} q_{s+2n},$$

$$p'_{s-2n} = (-)^n \frac{1}{\{i, s-2n+2\}\{i, s-2n+4\} \ldots \{i, s\}} q_{s-2n},$$

$$p'_{s+2n} = (-)^n \{i, s+2n\}\{i, s+2n-2\} \ldots \{i, s+2\} q_{s+2n}.$$

When $s = 0$, q_2/q_0 is equal to that which would be given by the general formula for $\frac{2q_{s+2n}}{q_{s+2n-2}}$ when we put in it $n = 1$, $s = 0$. Hence it follows that the q's for $s = 0$ have double the values given by the general formula.

If we change the sign of s, the two continued fractions in the equation for $\beta\sigma$ are simply interchanged. Hence $\beta\sigma$ is unchanged when s changes sign. Also, since $\{i, t\}$ is equal to $\{-i-1, t\}$, $\beta\sigma$ is unchanged when $-i-1$ is written for i. A consideration of the forms of the q's and p's shows that $q_{-s+2k}\,\mathrm{P}^{-s+2k}$ is equal to $\frac{i-s\,!}{i+s\,!} q_{s-2k}\,\mathrm{P}^{s-2k}$, and therefore

$$\left\{\begin{matrix}\mathfrak{P}_i^s \\ \mathsf{P}_i^s\end{matrix}\right. = \frac{i+s\,!}{i-s\,!}\left\{\begin{matrix}\mathfrak{P}_i^{-s} \\ \mathsf{P}_i^{-s}\end{matrix}\right.;$$

$$\left\{\begin{matrix}\mathfrak{P}_i^s \\ \mathsf{P}_i^s\end{matrix}\right. = \left\{\begin{matrix}\mathfrak{P}_{-i-1}^s \\ \mathsf{P}_{-i-1}^s\end{matrix}\right..$$

§ 7. *Rigorous determination of the Functions of the second degree.*

If a numerical value be attributed to β it is obviously possible to obtain the rigorous expressions for the several functions. Thus, if β were $\frac{1}{3}$ we could determine the harmonics of the ellipsoids of the class $c^2 = \frac{1}{2}(a^2 + b^2)$. But I do not think it is possible to obtain rigorous results in algebraic form when i is greater than 3. In order, however, to show how our formulæ lead to the required result I will determine the five functions corresponding to $i = 2$, but I will not work out the case of $i = 3$, although it is easy to do so.

When $s = 0$

$$\beta\sigma = \frac{\frac{1}{2}\beta^2\{2, 1\}\{2, 2\}}{4 + \beta\sigma} = \frac{12\beta^2}{4 + \beta\sigma}.$$

Therefore $\beta\sigma = -2 + 2(1 + 3\beta^2)^{\frac{1}{2}}$, or writing $B^2 = 1 + 3\beta^2$ for brevity, $\beta\sigma = 2(B - 1)$. Then putting $q_0 = 1$, and remembering that the value of q_2 is twice that given by the general formula,

$$q_2 = \frac{1}{4 + \beta\sigma} = \frac{B - 1}{6\beta^2}.$$

Therefore $$\mathfrak{P}_2 = \mathrm{P}_2 + \frac{B - 1}{6\beta}\mathrm{P}_2^{\,2} \quad . \quad . \quad . \quad . \quad (17),$$

where $\mathrm{P}_2 = \frac{3}{2}\nu^2 - \frac{1}{2}$, $\qquad \mathrm{P}_2^{\,2} = 3(\nu^2 - 1)$.

The coefficient of the cosine function is given by

$$p_2 = -\{2, 1\}\, q_2 = -6q_2.$$

Therefore $$\mathfrak{C}_2 = 1 - \frac{B - 1}{\beta}\cos 2\phi \quad . \quad . \quad . \quad . \quad (18).$$

$s = 1$, cosines; EOC type.

The continued fraction has $\frac{1}{4}\beta^2\{i, 2\}\{i, 3\}$ in the numerator, and vanishes because $\{2, 3\} = 0$. Therefore

$$\beta\sigma + \tfrac{1}{2}\beta i(i + 1) = 0, \text{ where } i = 2.$$

Therefore $$\sigma = -3.$$

But the coefficient is independent of β, for

$$\mathsf{P}_2^{\,1} = \Omega[q_1{}'\mathrm{P}_2^{\,1}], \text{ and } q_1{}' = 1.$$

Therefore $$\mathsf{P}_2^{\,1} = \sqrt{\frac{\nu^2 - \frac{1+\beta}{1-\beta}}{\nu^2 - 1}}\,\mathrm{P}_2^{\,1} = 3\nu\left(\nu^2 - \frac{1 + \beta}{1 - \beta}\right)^{\frac{1}{2}} \quad . \quad (19).$$

Clearly $$\mathsf{C}_2^{\,1} = \cos\phi\,(1 - \beta\cos 2\phi)^{\frac{1}{2}} \quad . \quad (20).$$

$s = 1$, sine; EOS type.

The continued fraction again vanishes and $\sigma = 3$, but it is not needed to express the functions. Putting $q_1' = 1$,

$$\mathfrak{P}_2^1 = q_1' \mathrm{P}_1^1 = \mathrm{P}_1^1 = 3\nu(\nu^2 - 1)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21),$$

$$\mathsf{S}_2^1 = \sin\phi\,(1 - \beta\cos 2\phi)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22).$$

$s = 2$, cosine; EEC type.

The second continued fraction vanishes because it contains $\{2, 3\}$ in the numerator. The equation is then

$$\beta\sigma = \frac{-\frac{1}{2}\beta^2\{2, 1\}\{2, 2\}}{4 - \beta\sigma} = \frac{-12\beta^2}{4 - \beta\sigma}.$$

Therefore $\beta\sigma = 2 - 2(1 + 3\beta^2)^{\frac{1}{2}} = 2(1 - B)$. Then putting $q_2 = 1$,

$$q_0 = -\tfrac{1}{2}\frac{\{2, 2\}\{2, 1\}}{4 - \beta\sigma} = \frac{-2(B - 1)}{\beta^2}.$$

Therefore $$\mathfrak{P}_2^2 = \frac{-2(B - 1)}{\beta}\mathrm{P}_2 + \mathrm{P}_2^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23),$$

where $$\mathrm{P}_2 = \tfrac{3}{2}\nu^2 - \tfrac{1}{2},$$
$$\mathrm{P}_2^2 = 3(\nu^2 - 1).$$

Then $$p_0 = -\frac{1}{\{2, 1\}}q_0 = -\tfrac{1}{6}q_0, \text{ and}$$

$$\mathfrak{C}_2^2 = \frac{B - 1}{3\beta} + \cos 2\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24).$$

$s = 2$, sine; EES type.

Both fractions disappear and σ vanishes, but is not needed for determining the functions. Noting that $q_0' = 0$, and $q_2' = 1$,

$$\mathsf{P}_2^2 = \Omega[q_2'\mathrm{P}_2^2] = 3\left(\nu^2 - \frac{1 + \beta}{1 - \beta}\right)^{\frac{1}{2}}(\nu^2 - 1)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad (25),$$

$$\mathfrak{S}_2^2 = \sin 2\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26).$$

We can write down the functions of μ by symmetry, and the products of the three functions give rigorously the five solid harmonic solutions of LAPLACE'S equation of the second degree. As I have remarked above, the seven harmonics of the third degree may be obtained rigorously by a parallel process.

§ 8. *Approximate Form of the Functions.*

It is clear that the first approximation to $\beta\sigma$ is zero, and that the second approximation, in the general case, is

$$\beta\sigma = -\tfrac{1}{16}\beta^2 \frac{\{i, s\}\{i, s-1\}}{s-1} + \tfrac{1}{16}\beta^2 \frac{\{i, s+1\}\{i, s+2\}}{s+1}$$

$$= \tfrac{1}{8}\beta^2 \left(3s^2 - 2i(i+1) - \frac{i^2(i+1)^2}{s^2-1}\right).$$

If this expression were inserted in $\frac{q_{s\pm 2}}{q_s}$ we should obtain $q_{s\pm 2}$ correct to β^2. But since the next approximation would only introduce β^4, it follows that $q_{s\pm 2}$ would be correct to β^3 inclusive. Now $q_{s\pm 2}$ enters in the functions with a factor β, and therefore this approximation would give results correct to β^4 inclusive. Since the similar operation could be applied with equal ease in all the cases in which the continued fractions assume special forms, it follows that this degree of accuracy is very easily attainable. However, the forms of the coefficients would be rather complicated, and it would render the subsequent algebra so tedious that I do not propose at present to carry the approximation beyond β^2.

It now suffices to put $\sigma = 0$ in the denominators of all the continued fractions, whereby the coefficients are determined, except in the cases of $s = 1$, $s = 3$, where we put $\sigma = \pm \frac{1}{2}i(i+1)$.

In the general case we have

$$\left.\begin{aligned}
q_{s-2} &= -\tfrac{1}{8}\frac{\{i, s\}\{i, s-1\}}{s-1}, & q_{s+2} &= \frac{1}{8(s+1)}, \\
q_{s-4} &= \frac{\{i, s\}\{i, s-1\}\{i, s-2\}\{i, s-3\}}{128(s-1)(s-2)}, & q_{s+4} &= \frac{1}{128(s+1)(s+2)}, \\
q'_{s-2} &= \frac{s-2}{s}\, q_{s-2}, & q'_{s+2} &= \frac{s+2}{s}\, q_{s+2}, \\
q'_{s-4} &= \frac{s-4}{s}\, q_{s-4}, & q'_{s+4} &= \frac{s+4}{s}\, q_{s+4}, \\
p_{s-2} &= \frac{\{i, s\}}{8(s-1)}, & p_{s+2} &= \frac{-\{i, s+1\}}{8(s+1)}, \\
p_{s-4} &= \frac{\{i, s\}\{i, s-2\}}{128(s-1)(s-2)}, & p_{s+4} &= \frac{\{i, s+1\}\{i, s+3\}}{128(s+1)(s+2)}, \\
p'_{s-2} &= \frac{\{i, s-1\}}{8(s-1)}, & p'_{s+2} &= \frac{-\{i, s+2\}}{8(s+1)}, \\
p'_{s-4} &= \frac{\{i, s-1\}\{i, s-3\}}{128(s-1)(s-2)}, & p'_{s+4} &= \frac{\{i, s+2\}\{i, s+4\}}{128(s+1)(s+2)}
\end{aligned}\right\} \quad . \quad (27).$$

When $s = 0$, we double the results given by the general formula and find

$$\left.\begin{aligned}
&q_2 = \tfrac{1}{4},\ q_4 = \tfrac{1}{128},\ p_2 = -\tfrac{1}{4}\{i, 1\},\ p_4 = \tfrac{1}{128}\{i, 1\}\{i, 3\}. \\
\text{There are no } &q_2',\ q_4',\ \text{and } p_2' = -\tfrac{1}{4}\{i, 2\},\ p_4' = \tfrac{1}{128}\{i, 2\}\{i, 4\}
\end{aligned}\right\} \quad . \quad . \quad (28).$$

When $s = 1$,

$$q_3 = \frac{1}{16 \mp \beta i(i+1)} = \tfrac{1}{16}(1 \pm \tfrac{1}{16}\beta i(i+1)),$$ with upper sign for cosines (EOC, OOC) and lower sign for sines (OOS, EOS).

$$q_5 = \frac{1}{128 \cdot 2 \cdot 3} = \frac{1}{768} \text{ for all cases.}$$

But for OOS, EOS we use the 𝔓 form, and for EOC, OOC the P form; and for the latter $\frac{s+2}{s} = 3$, $\frac{s+4}{s} = 5$.

Therefore for OOS, EOS (sines)

$$q_3 = \tfrac{1}{16}(1 - \tfrac{1}{16}\beta i(i+1)), \quad q_5 = \tfrac{1}{768},$$

and for EOC, OOC (cosines) . . . . . . (29).

$$q_3' = \tfrac{3}{16}(1 + \tfrac{1}{16}\beta i(i+1)), \quad q_5' = \tfrac{5}{768}$$

For OOC, OOS, with upper sign for cosine and lower sign for sine,

$$p_3 = -\tfrac{1}{16}\{i, 2\}[1 \pm \tfrac{1}{16}\beta i(i+1)], \quad p_5 = \tfrac{1}{768}\{i, 2\}\{i, 4\} \quad . \quad . \ (29).$$

For EOC, EOS, with upper sign for cosine and lower sign for sine,

$$p_3' = -\tfrac{1}{16}\{i, 3\}[1 \pm \tfrac{1}{16}\beta i(i+1)], \quad p_5' = \tfrac{1}{768}\{i, 3\}\{i, 5\} \quad . \quad . \ (29).$$

When $s = 2$ the coefficients may be derived from the general formula.

When $s = 3$

$$q_1 = \frac{-\frac{1}{2}\{i, 3\}\{i, 2\}}{8 \mp \frac{1}{2}\beta i(i+1)} = -\tfrac{1}{16}\{i, 2\}\{i, 3\}[1 \pm \tfrac{1}{16}\beta i(i+1)],$$

the upper sign applying to cosines (OOC, EOC) the lower to sines (OOS, EOS);

$$q_5 = \tfrac{1}{32}, \quad q_7 = \tfrac{1}{2560}.$$

But for OOS, EOS the 𝔓 form applies, and for OOC, EOC the P form applies.

Also with $s = 3$, $\frac{s-2}{s} = \frac{1}{3}$, $\frac{s+2}{s} = \frac{5}{3}$, $\frac{s+4}{s} = \frac{7}{3}$.

Therefore for OOS, EOS

$$q_1 = -\tfrac{1}{16}\{i, 2\}\{i, 3\}[1 - \tfrac{1}{16}\beta i(i+1)],\ q_5 = \tfrac{1}{32},\ q_7 = \tfrac{1}{2560}.$$

For OOC, EOC

$$q_1' = -\tfrac{1}{48}\{i, 2\}\{i, 3\}[1 + \tfrac{1}{16}\beta i(i+1)],\ q_5' = \tfrac{5}{96},\ q_7' = \tfrac{7}{7680}.$$

For OOC, OOS, with upper sign for cosine and lower for sine,

$$p_1 = \tfrac{1}{16}\{i, 3\}[1 \pm \tfrac{1}{16}\beta i(i+1)],\ p_5 = -\tfrac{1}{32}\{i, 4\},\quad p_7 = \tfrac{1}{2560}\{i, 4\}\{i, 6\}.$$

For EOC, EOS, with upper sign for cosine and lower for sine,

$$p_1' = \tfrac{1}{16}\{i, 2\}[1 \pm \tfrac{1}{16}\beta i(i+1)],\ p_5' = -\tfrac{1}{32}\{i, 5\},\quad p_7' = \tfrac{1}{2560}\{i, 5\}\{i, 7\} \qquad (30).$$

It will save much trouble to note that if we were to admit negative suffixes to the q's, the general formula would give us the term $\beta^2 q_{-1}\,\mathrm{P}^{-1}$, where

$$q_{-1} = \frac{\{i\,3\}\{i, 2\}\{i, 1\}\{i, 0\}}{128 . 2 . 1}.$$

Thus this term is $\frac{1}{(16)^2}\beta^2 i(i+1) . \{i, 3\}\{i, 2\}\mathrm{P}^1$. But this is exactly that part of the term in (30) which arises from $\beta q_1 \mathrm{P}^1$, but which is not included in the general formula.

Similarly the general formula gives for q'_{-1}, p_{-1}, p'_{-1} those parts of the terms arising from q_1', p_1, p_1' which are not included in the general formula.

It follows that in much of the subsequent work we need not devote special consideration to the case of $s = 3$.

§ 9. *Factors of Transformation between the two forms of P-function and C- or S-function.*

The rigorous expressions $\mathfrak{P}^s$ and P^s always differ from one another, but approximately they are the same up to a certain power of β, provided that s is greater than a certain quantity.

Since $\Omega = \left(\frac{\nu^2 - \frac{1+\beta}{1-\beta}}{\nu^2 - 1}\right)^{\frac{1}{2}} = \left(1 + \frac{2\beta}{(\nu^2 - 1)(1-\beta)}\right)^{\frac{1}{2}}$, it is legitimate to develop Ω in powers of $1/(\nu^2 - 1)$ up to a certain power, say t, provided that it is to be multiplied by a function involving at least $(\nu^2 - 1)^t$ as a factor; for this condition insures that there shall be no infinite terms when $\nu = \pm 1$. At present, I limit the development to β^2, so that

$$\Omega = 1 - \frac{\beta + \beta^2}{\nu^2 - 1} - \frac{\frac{1}{2}\beta^2}{(\nu^2 - 1)^2}.$$

Therefore

$$\mathbf{P}^s = \left(1 - \frac{\beta + \beta^2}{\nu^2 - 1} - \frac{\frac{1}{2}\beta^2}{(\nu^2 - 1)^2}\right)\mathrm{P}^s + \left(1 - \frac{\beta}{\nu^2 - 1}\right)(\beta q'_{s-2}\mathrm{P}^{s-2} + \beta q'_{s+2}\mathrm{P}^{s+2}) + \beta^2 q'_{s-4}\mathrm{P}^{s-4} + \beta^2 q'_{s+4}\mathrm{P}^{s+4}.$$

It is obvious on inspection that we cannot rely on this development if s is less than 4.

If then s is equal to, or greater than 4, this value of $\mathbf{P}^s$, when properly developed, to the adopted order of approximation can only differ from $\mathfrak{P}^s$ by a constant factor, say C_i^s or shortly C^s; so that

$$\mathfrak{P}^s = C^s \mathbf{P}^s \quad \ldots \ldots \ldots \quad (31),$$

and we have to determine the constant C^s.

We might develop the above expression for $\mathbf{P}^s$ completely and compare it with $\mathfrak{P}^s$, but this is unnecessary since the comparison of a single term suffices.

I now write

$$\Sigma_i^s = \frac{i(i+1)}{s^2 - 1} \quad \ldots \ldots \ldots \quad (32),$$

or shortly Σ. This notation is introduced because this function occurs very frequently hereafter.

We have seen in (11) (slightly modified) that

$$\frac{\mathrm{P}^s}{\nu^2 - 1} = \tfrac{1}{4}\left\{\frac{\mathrm{P}^{s+2}}{s(s+1)} - 2(\Sigma + 1)\mathrm{P}^s + \frac{\{i, s\}\{i, s-1\}}{s(s-1)}\mathrm{P}^{s-2}\right\}.$$

We may write this

$$\frac{\mathrm{P}^s}{\nu^2 - 1} = \alpha_s \mathrm{P}^{s+2} + \beta_s \mathrm{P}^{s+1} + \gamma_s \mathrm{P}^{s-2},$$

where $\alpha_s = \dfrac{1}{4s(s+1)}$, $\beta_s = -\tfrac{1}{2}(\Sigma + 1)$, $\gamma_s = \dfrac{\{i, s\}\{i, s-1\}}{4s(s-1)}$.

Then

$$\frac{\mathrm{P}^s}{(\nu^2 - 1)^2} = \alpha_s(\alpha_{s+2}\mathrm{P}^{s+4} + \beta_{s+2}\mathrm{P}^{s+2} + \gamma_{s+2}\mathrm{P}^s) + \beta_s(\alpha_s \mathrm{P}^{s+2} + \beta_s \mathrm{P}^s + \gamma_s \mathrm{P}^{s-2}) + \gamma_s(\alpha_{s-2}\mathrm{P}^s + \beta_{s-2}\mathrm{P}^{s-2} + \gamma_{s-2}\mathrm{P}^{s-4}).$$

Therefore the coefficient of P^s is $\alpha_s\gamma_{s+2} + (\beta_s)^2 + \alpha_{s-2}\gamma_s$, or

$$\tfrac{1}{16}\left[\frac{\{i, s+2\}\{i, s+1\}}{s(s+1)^2(s+2)} + 4(\Sigma + 1)^2 + \frac{\{i, s\}\{i, s-1\}}{s(s-1)^2(s-2)}\right].$$

I now introduce a further abridgement and write

$$\Upsilon_i^s = \frac{(i-1)\,i\,(i+1)\,(i+2)}{s^2-4} \quad \ldots \ldots \ldots \quad (32),$$

or shortly Υ.

Then, after reduction, I find

$$\frac{\mathrm{P}^s}{(\nu^2-1)^2} = \tfrac{1}{8}[-\Sigma^2 s^2 + \Sigma^2 + 4\Sigma + 3 + \Upsilon]\,\mathrm{P}^s + \ldots$$

Accordingly the coefficient of P^s in $\mathbf{P}^s$ is

$$1 + \tfrac{1}{2}\beta(\Sigma+1) + \tfrac{1}{2}\beta^2(\Sigma+1) - \tfrac{1}{16}\beta^2[-\Sigma^2 s^2 + \Sigma^2 + 4\Sigma + 3 + \Upsilon]$$
$$- \tfrac{1}{4}\beta^2\left[\frac{q'_{s-2}}{(s-1)(s-2)} + q'_{s+2}\frac{\{i, s+1\}\{i, s+2\}}{(s+1)(s+2)}\right].$$

But $q'_{s-2} = -\dfrac{(s-2)\{i, s\}\{i, s-1\}}{8s(s-1)}$, $q'_{s+2} = \dfrac{s+2}{8s(s+1)}$, and the last term in the above expression will be found to be equal to $+\tfrac{1}{8}\beta^2(\Sigma^2-1)$. Thus the coefficient of P^s in the development of $\mathbf{P}^s$ is

$$1 + \tfrac{1}{2}\beta(\Sigma+1) + \tfrac{1}{16}\beta^2(\Sigma^2 s^2 + \Sigma^2 + 4\Sigma + 3 - \Upsilon);$$

but the same coefficient in $\mathfrak{P}^s$ is unity.

Therefore

$$\left.\begin{aligned}
\frac{1}{\mathrm{C}_i^s} &= 1 + \tfrac{1}{2}\beta(\Sigma+1) + \tfrac{1}{16}\beta^2(\Sigma^2 s^2 + \Sigma^2 + 4\Sigma + 3 - \Upsilon) \\
\mathrm{C}_i^s &= 1 - \tfrac{1}{2}\beta(\Sigma+1) + \tfrac{1}{16}\beta^2(-\Sigma^2 s^2 + 3\Sigma^2 + 4\Sigma + 1 + \Upsilon) \\
\frac{1}{(\mathrm{C}_i^s)^2} &= 1 + \beta(\Sigma+1) + \tfrac{1}{8}\beta^2(\Sigma^2 s^2 + 3\Sigma^2 + 8\Sigma + 5 - \Upsilon) \\
(\mathrm{C}_i^s)^2 &= 1 - \beta(\Sigma+1) + \tfrac{1}{8}\beta^2(-\Sigma^2 s^2 + 5\Sigma^2 + 8\Sigma + 3 + \Upsilon)
\end{aligned}\right\} \quad . \quad . \ (33).$$

The squares of this constant and of its reciprocal are given because they will be needed at a later stage.

We next consider the cosine and sine functions.

$$\left\{\begin{matrix}\mathbf{C}^s \\ \mathbf{S}^s\end{matrix}\right. = \Phi\left[\left\{\begin{matrix}\cos \\ \sin\end{matrix}\right. s\phi + \beta p'_{s-2}\left\{\begin{matrix}\cos \\ \sin\end{matrix}\right.(s-2)\phi + \ldots\right].$$

As far as β^2

$$\Phi = (1 - \beta\cos 2\phi)^{\frac{1}{2}} = 1 - \tfrac{1}{2}\beta\cos 2\phi - \tfrac{1}{16}\beta^2(1 + \cos 4\phi).$$

Therefore

$$\begin{Bmatrix}\mathbf{C}^s\\ \mathbf{S}^s\end{Bmatrix} = [1 - \tfrac{1}{2}\beta\cos 2\phi - \tfrac{1}{16}\beta^2(1+\cos 4\phi)]\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}s\phi$$
$$+\,\beta\left[p'_{s-2}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(s-2)\phi + p'_{s+2}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(s+2)\phi\right][1-\tfrac{1}{2}\beta\cos 2\phi]$$
$$+\,\beta^2 p'_{s-4}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(s-4)\phi + \beta^2 p'_{s+4}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(s+4)\phi.$$

This expression, when developed, must lead to $\mathfrak{C}^s$ or $\mathfrak{S}^s$ multiplied by a constant factor.

Let
$$\begin{Bmatrix}\mathfrak{C}^s\\ \mathfrak{S}^s\end{Bmatrix} = \mathrm{D}_i^s\begin{Bmatrix}\mathbf{C}^s\\ \mathbf{S}^s\end{Bmatrix} \quad . \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;. \quad (34).$$

Then D_i^s or D^s may be found by considering only the coefficient of $\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}s\phi$. Hence

$$\frac{1}{\mathrm{D}^s} = 1 - \tfrac{1}{16}\beta^2 - \tfrac{1}{4}\beta^2 p'_{s-2} - \tfrac{1}{4}\beta^2 p'_{s+2}.$$

But
$$p'_{s-2} = \frac{\{i, s-1\}}{8(s-1)}, \qquad p'_{s+2} = \frac{-\{i, s+2\}}{8(s+1)}, \qquad \text{and}$$
$$p'_{s-2} + p'_{s+2} = \tfrac{1}{4}(\Sigma + 2).$$

Therefore

$$\left.\begin{aligned}\frac{1}{\mathrm{D}_i^s} &= 1 - \tfrac{1}{16}\beta^2(\Sigma+3),\\ \frac{1}{(\mathrm{D}_i^s)^2} &= 1 - \tfrac{1}{8}\beta^2(\Sigma+3)\end{aligned}\right\} \quad . \;.\;.\;.\;.\;.\;.\;.\;. \quad (35).$$

The reciprocals may clearly be written down at once.

There are no factors by which $\mathfrak{P}^3$, $\mathfrak{P}^2$, $\mathfrak{P}^1$ can be converted into $\mathbf{P}^3$, $\mathbf{P}^2$, $\mathbf{P}^1$; but this is not true of the cosine and sine functions.

In the case of $s = 3$, it will be found that the general formula holds good for the factor whereby $\mathfrak{C}^3$, $\mathfrak{S}^3$ are convertible into $\mathbf{C}^3$, $\mathbf{S}^3$.

When $s = 2$,

$$\begin{Bmatrix}\mathbf{C}^2\\ \mathbf{S}^2\end{Bmatrix} = [1 - \tfrac{1}{2}\beta\cos 2\phi - \tfrac{1}{16}\beta^2(1+\cos 4\phi)]\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}2\phi + \beta p'_0(1-\tfrac{1}{2}\beta\cos 2\phi)\begin{Bmatrix}1\\ 0\end{Bmatrix}$$
$$+\,\beta p'_4(1-\tfrac{1}{2}\beta\cos 2\phi)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}4\phi + \beta^2 p'_6 \begin{matrix}\cos\\ \sin\end{matrix} 6\phi.$$

Then

$$\mathbf{C}^2 = [1 - (\tfrac{3}{32} + \tfrac{1}{2}p'_0 + \tfrac{1}{4}p'_4)\beta^2]\cos 2\phi + \ldots$$
$$\mathbf{S}^2 = [1 - (\tfrac{1}{32} + \tfrac{1}{4}p'_4)\beta^2]\sin 2\phi + \ldots$$

But $p'_0 = \frac{1}{8}\{i, 1\}$, $p'_4 = -\frac{1}{24}\{i, 4\}$, $p'_6 = \frac{1}{1536}\{i, 4\}\{i, 6\}$, and

$$\tfrac{3}{32} + \tfrac{1}{2}p'_0 + \tfrac{1}{4}p'_4 = \tfrac{1}{32}(5\Sigma + 7),$$
$$\tfrac{1}{32} + \tfrac{1}{4}p'_4 \qquad = -\tfrac{1}{32}(\Sigma - 5), \quad \text{where } \Sigma = \frac{i(i+1)}{2^2 - 1} = \tfrac{1}{3}i(i+1).$$

Therefore the factors are

$$\left.\begin{aligned} \frac{1}{D_i^2}(\cos) &= 1 - \tfrac{1}{32}\beta^2(5\Sigma^2 + 7), \\ \frac{1}{D_i^2}(\sin) &= 1 + \tfrac{1}{32}\beta^2(\Sigma^2 - 5) \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; (36).$$

It is easy to verify that the other coefficients of $\mathfrak{C}^2$ and $\mathfrak{S}^2$ are in fact reproduced.

The notation adopted here and below for distinguishing the cosine and sine factors is perhaps rather clumsy, but I have not thought it worth while to take distinctive symbols for the factors in these cases, because they will not be of frequent occurrence. When $s = 1$,

$$\begin{Bmatrix} \mathsf{C}^1 \\ \mathsf{S}^1 \end{Bmatrix} = [1 - \tfrac{1}{2}\beta\cos 2\phi - \tfrac{1}{16}\beta^2(1 + \cos 4\phi)]\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\phi + \beta p'_3(1 - \tfrac{1}{2}\beta\cos 2\phi)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}3\phi$$
$$+ \beta^2 p'_5 \begin{Bmatrix}\cos\\ \sin\end{Bmatrix}5\phi$$
$$= [1 \mp \tfrac{1}{4}\beta - \tfrac{1}{4}\beta^2(p'_3 + \tfrac{1}{4})]\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\phi + \dots$$

This must be equal to $\dfrac{1}{D_i^1}\begin{Bmatrix}\mathfrak{C}^1\\ \mathfrak{S}^1\end{Bmatrix}$.

Now, with upper sign for cosine and lower for sine,

$$p'_3 = -\tfrac{1}{16}\{i, 3\}[1 \pm \tfrac{1}{16}\beta i(i+1)], \quad p'_5 = \tfrac{1}{768}\{i, 3\}\{i, 5\}.$$

Substituting for p'_3 its values, we find with the upper sign

$$\frac{1}{D_i^1}(\cos) = 1 - \tfrac{1}{4}\beta - \tfrac{1}{4}\beta^2(p'_3 + \tfrac{1}{4}) = 1 - \tfrac{1}{4}\beta + \tfrac{1}{64}\beta^2[i(i+1) - 10].$$

And with the lower sign

$$\frac{1}{D_i^1}(\sin) = 1 + \tfrac{1}{4}\beta - \tfrac{1}{4}\beta^2(p'_3 + \tfrac{1}{4}) = 1 + \tfrac{1}{4}\beta + \tfrac{1}{64}\beta^2[i(i+1) - 10] \qquad (37).$$

It follows that

$$\left.\begin{aligned} D_i^1(\cos) &= 1 + \tfrac{1}{4}\beta - \tfrac{1}{64}\beta^2[i(i+1) - 14] \\ D_i^1(\sin) &= 1 - \tfrac{1}{4}\beta - \tfrac{1}{64}\beta^2[i(i+1) - 14] \\ \left[\frac{1}{D_i^1}(\cos)\right]^2 &= 1 - \tfrac{1}{2}\beta + \tfrac{1}{32}\beta^2[i(i+1) - 8] \\ [D_i^1(\cos)]^2 &= 1 + \tfrac{1}{2}\beta - \tfrac{1}{32}\beta^2[i(i+1) - 16] \\ \left[\frac{1}{D_i^1}(\sin)\right]^2 &= 1 + \tfrac{1}{2}\beta + \tfrac{1}{32}\beta^2[i(i+1) - 8] \\ [D_i^1(\sin)]^2 &= 1 - \tfrac{1}{2}\beta - \tfrac{1}{32}\beta^2[i(i+1) - 16] \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; (37).$$

We cannot in the present case use Σ_i^1 as an abridged notation, because it is infinite as involving $s^2 - 1$ in the denominator.

It is easy to verify that the other coefficients of $\mathfrak{C}^1$ and $\mathfrak{S}^1$ are, in fact, reproduced in the transformation.

Lastly when $s = 0$, we have only cosine functions. As before

$$\mathsf{C} = 1 - \tfrac{1}{2}\beta\cos 2\phi - \tfrac{1}{16}\beta^2(1 + \cos 4\phi) + \beta p'_2(1 - \tfrac{1}{2}\beta\cos 2\phi)\cos 2\phi + \beta^2 p'_4 \cos 4\phi$$

$$= 1 - \tfrac{1}{16}\beta^2 - \tfrac{1}{4}\beta^2 p'_2 + \ldots$$

This must be equal to $\dfrac{\mathfrak{C}}{D_i}$, and therefore $\dfrac{1}{D_i} = 1 - \tfrac{1}{16}\beta^2 - \tfrac{1}{4}\beta^2 p'_2$.

Now $\qquad p'_2 = -\tfrac{1}{4}\{i, 2\}, \quad p'_4 = \tfrac{1}{128}\{i, 2\}\{i, 4\}.$

Hence
$$\frac{1}{D_i} = 1 + \tfrac{1}{16}\beta^2[i(i+1) - 3] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38).$$

Since in this case $\Sigma_i = \dfrac{i(i+1)}{-1}$,

$$\frac{1}{D_i} = 1 - \tfrac{1}{16}\beta^2[\Sigma_i + 3] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38).$$

Thus the general formula again holds good.

It is easy to verify that the other coefficients of $\mathfrak{C}$ are in fact reproduced.

The principal use of the transforming factors, determined in this section, is that it will enable us to avoid some tedious analysis hereafter.

§ 10. *The Functions of the Second Kind.*

The second continued fraction of § 6 terminates because

$$\{i, s + 2n + 2\}\{i, s + 2n + 1\} = 0$$

when $n = \frac{1}{2}(i - s)$ or $\frac{1}{2}(i - s - 1)$, since one of the two factors then assumes the form $\{i, i + 1\}$.

Hence it follows that the equation for determining σ is the same as before; but we cannot on that account assume that the q coefficients vanish when their suffixes are greater than i.

In considering the P-functions it was immaterial whether or not we regarded them as vanishing, because P^t vanishes if t is greater than i. But the Q-functions do not vanish in this case, and therefore we must postulate the existence of q's with suffix greater than i.

In fact, whilst we have as before, when i and s are both odd or both even,

$$\frac{2q_i}{q_{i-2}} = \frac{1}{i^2 - s^2 + \beta\sigma},$$

we also have

$$\frac{2q_{i+2}}{q_i} = \frac{1}{(i+2)^2 - s^2 + \beta\sigma - \frac{1}{4}\beta^2 \{i, i+4\}\{i, i+3\}\left(\frac{2q_{i+4}}{q_{i+2}}\right)},$$

and similarly a fraction for $\frac{2q_{i+4}}{q_{i+2}}$, and so forth.

It follows therefore that while the q's with suffixes less than or equal to i depend on finite continued fractions, those with suffixes greater than i depend on infinite continued fractions.

It thus appears that while the first series in the expression for $\mathfrak{Q}_i^s$ or for $\mathbf{Q}_i^s$ has limits 1 to $\frac{1}{2}s$ or $\frac{1}{2}(s-1)$, as before, the limits of the second series are 1 to ∞.

Thus we have found an expansion for this class of functions in powers of β.

In the limited case in which the coefficients have been actually evaluated, namely, where the development is only carried as far as the squares of β, we have

$$q_{s+2} = \frac{1}{8(s+1)}, \qquad q_{s+4} = \frac{1}{128(s+1)(s+2)},$$

$$q'_{s+2} = \frac{s+2}{8s(s+1)}, \qquad q'_{s+4} = \frac{s+4}{128s(s+1)(s+2)}.$$

These coefficients do not vanish when $s+2$ or $s+4$ are greater than i, and this confirms the conclusion already arrived at.

In spherical harmonic analysis there is no occasion to consider the value of Q_i^s when s is greater than i, and the values are therefore not familiar. I will therefore now determine them.

It is known* that

$$Q_i = \frac{2^i (i\,!)^2}{2i+1\,!}\left[\frac{1}{\nu^{i+1}} + \frac{i+2\,!}{2\,.\,1\,!\,i\,!}\cdot\frac{1}{(2i+3)\nu^{i+3}} + \frac{i+4\,!}{2^2\,.\,2\,!\,i\,!}\,\frac{1}{(2i+3)(2i+5)\nu^{i+5}} + \ldots\right].$$

Therefore differentiating

$$\left(\frac{d}{d\nu}\right)^{i+1} Q_i = (-)^{i+1} 2^i\,.\,i\,!\left[\frac{1}{\nu^{2i+2}} + \frac{i+1}{1\,!}\,\frac{1}{\nu^{2i+4}} + \frac{(i+1)(i+2)}{2\,!}\cdot\frac{1}{\nu^{2i+6}} + \ldots\right],$$

$$= (-)^{i+1}\frac{2^i\,.\,i\,!}{(\nu^2-1)^{i+1}}.$$

And

$$\left(\frac{d}{d\nu}\right)^{i+2} Q_i = (-)^i \frac{2^{i+1}\,.\,i+1\,!}{(\nu^2-1)^{i+2}}\,\nu.$$

* BRYAN, 'Camb. Phil. Soc. Proc.,' vol. 6, 1888, p. 293.

$$\left(\frac{d}{d\nu}\right)^{i+3} Q_i = (-)^i 2^{i+1} . \, \overline{i+1}\,! \left[\frac{-2(i+2)\nu^2}{(\nu^2-1)^{i+3}} + \frac{1}{(\nu^2-1)^{i+2}}\right],$$

$$= (-)^{i+1}\left[\frac{2^{i+2}.\,\overline{i+2}\,!}{(\nu^2-1)^{i+3}} + \frac{2^{i+1}(2i+3).\,\overline{i+1}\,!}{(\nu^2-1)^{i+2}}\right].$$

$$\left(\frac{d}{d\nu}\right)^{i+4} Q_i = (-)^i\left[\frac{2^{i+3}.\,\overline{i+3}\,!}{(\nu^2-1)^{i+4}} + \frac{2^{i+2}(2i+3).\,\overline{i+2}\,!}{(\nu^2-1)^{i+3}}\right]\nu.$$

But $$Q^t = (\nu^2-1)^{\frac{1}{2}t}\left(\frac{d}{d\nu}\right)^t Q_i,$$

therefore
$$\left.\begin{aligned}
Q_i^{i+1} &= (-)^{i+1}\frac{2^i\, i\,!}{(\nu^2-1)^{\frac{1}{2}(i+1)}} \\
Q_i^{i+2} &= (-)^i\frac{2^{i+1}.\,\overline{i+1}\,!\,\nu}{(\nu^2-1)^{\frac{1}{2}(i+2)}} \\
Q_i^{i+3} &= (-)^{i+1}\frac{2^{i+1}.\,\overline{i+1}\,!}{(\nu^2-1)^{\frac{1}{2}(i+3)}}[2i+4+(2i+3)(\nu^2-1)] \\
Q_i^{i+4} &= (-)^i\frac{2^{i+2}.\,\overline{i+2}\,!\,\nu}{(\nu^2-1)^{\frac{1}{2}(i+4)}}[2i+6+(2i+3)(\nu^2-1)]
\end{aligned}\right\} \quad . \quad . \quad (39).$$

These are all the functions which can be needed for the expression as far as β^2 of $\mathfrak{Q}_i^s$ or of Q_i^s when s is less or equal to i. If s is equal to i, we shall have terms $\beta^2\, q_{i+4}\, Q^{s+4}$ or $\Omega\beta^2\, q'_{s+4}\, Q^{s+4}$, and these are the furthest.

But it is well known that there is another expression for these functions of the second kind.

The differential equation is

$$\left[(\nu^2-1)^2\frac{d^2}{d\nu^2} + 2\nu(\nu^2-1)\frac{d}{d\nu} - i(i+1)(\nu^2-1) - s^2\right]\mathfrak{Q}_i^s$$
$$- \beta\left[(\nu^2-1)(\nu^2+1)\frac{d^2}{d\nu^2} + 2\nu^3\frac{d}{d\nu} - i(i+1)\nu^2 - \sigma\right]\mathfrak{Q}_i^s = 0,$$

where $\mathfrak{Q}_i^s$ may be interpreted as meaning also Q_i^s.

Let us assume that

$$\mathfrak{Q}^s = \mathfrak{P}^s\int_\nu^\infty V\,d\nu$$

is a solution, where $\mathfrak{Q}^s$, $\mathfrak{P}^s$ may be interpreted as meaning also Q^s, P^s.

Then since $\mathfrak{P}^s$ is a solution of the differential equation, we have

$$(\nu^2-1)^2\left[2V\frac{d\mathfrak{P}^s}{d\nu} + \mathfrak{P}^s\frac{dV}{d\nu}\right] + 2\nu(\nu^2-1)\mathfrak{P}^s V$$
$$- \beta\left[(\nu^2-1)(\nu^2+1)\left(2V\frac{d\mathfrak{P}^s}{d\nu} + \mathfrak{P}^s\frac{dV}{d\nu}\right) + 2\nu^3\mathfrak{P}^s V\right] = 0.$$

This is easily reducible to

$$\frac{d}{d\nu}\log\left[\mathrm{V}\,(\mathfrak{P}^s)^2(\nu^2-1)^{\frac{1}{2}}\left(\nu^2-\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}\right]=0,$$

whence $\mathrm{V}=\dfrac{\mathfrak{E}_i^s}{(\mathfrak{P}^s)^2(\nu^2-1)^{\frac{1}{2}}\left(\nu^2-\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}}$, where $\mathfrak{E}_i^s$ is a constant.

Hence

$$\left.\begin{aligned}\mathfrak{Q}_i^s&=\mathfrak{E}_i^s\mathfrak{P}_i^s\int_\nu^\infty\frac{d\nu}{(\mathfrak{P}_i^s)^2(\nu^2-1)^{\frac{1}{2}}\left(\nu^2-\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}},\\ \mathsf{Q}_i^s&=\mathsf{E}_i^s\mathsf{P}_i^s\int_\nu^\infty\frac{d\nu}{(\mathsf{P}_i^s)^2(\nu^2-1)^{\frac{1}{2}}\left(\nu^2-\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}}\end{aligned}\right\}\quad\ldots\ldots\ (40).$$

The general solution of the differential equation must be

$$\alpha\,\mathfrak{P}_i^s+\gamma\,\mathfrak{Q}_i^s,$$

and we have already found both $\mathfrak{P}^s$ and $\mathfrak{Q}^s$. Hence the two $\mathfrak{Q}^s$'s must be different expressions for the same thing, for the form of $\mathfrak{Q}^s$ as a series negatives the hypothesis that it involves $\mathfrak{P}^s$ in the form $\gamma_1\mathfrak{P}^s+\gamma_2\mathfrak{Q}^s$.

Having then two forms of $\mathfrak{Q}^s$ or of Q^s, it remains to evaluate the coefficients $\mathfrak{E}_i^s$, E_i^s, which are involved in the equations (40). In order to do this it will suffice to consider the case where ν is very great, so that

$$\mathrm{P}^t=\frac{2i\,!}{2^i\,i\,!}\cdot\frac{\nu^i}{i-t\,!},\qquad \mathrm{Q}^t=(-)^t\,\frac{2^i\,.\,i\,!}{2i+1\,!}\,\frac{i+t\,!}{\nu^{i+1}}.$$

As far as concerns the first term in the series

$$\mathfrak{P}^s=\frac{2i\,!}{2^i i\,!}\,\frac{\nu^i}{i-s\,!}\left[1+\beta q_{s-2}\frac{i-s\,!}{i-s+2\,!}+\beta q_{s+2}\frac{i-s\,!}{i-s-2}\right.$$
$$\left.+\beta^2 q_{s-4}\frac{i-s\,!}{i-s+4\,!}+\beta^2 q_{s+4}\frac{i-s\,!}{i-s-4\,!}\right]$$

$$\mathfrak{Q}^s=(-)^s\,\frac{2^i\,.\,i\,!}{2i+1\,!}\,\frac{i+s\,!}{\nu^{i+1}}\left[1+\beta q_{s-2}\frac{i+s-2\,!}{i+s\,!}+\beta q_{s+2}\frac{i+s+2\,!}{i+s\,!}\right.$$
$$\left.+\beta^2 q_{s-4}\frac{i+s-4\,!}{i+s\,!}+\beta^2 q_{s+4}\frac{i+s+4\,!}{i+s\,!}\right].$$

It will be observed that if s is equal to i or $i-1$ the terms in $\mathfrak{P}^s$ in q_{s+2} and q_{s+4} disappear; and if s is equal to $i-2$ or $i-3$ that in q_{s+4} disappears. This agrees, as it should do, with the vanishing of P^{s+2} and P^{s+4} when the order is greater than the degree.

If we write $\mathfrak{P}^s = \alpha\nu^i$ and $\mathfrak{Q}^s = \frac{\gamma}{\nu^{i+1}}$, the first of our equations (40) becomes, when ν is very large,

$$\frac{\gamma}{\nu^{i+1}} = \mathfrak{E}^s \alpha\nu^i \int_\nu^\infty \frac{d\nu}{\nu^2(\alpha\nu^i)^2} = \mathfrak{E}^s \frac{\nu^i}{\alpha} \int_\nu^\infty \frac{d\nu}{\nu^{2i+2}}$$

$$= \frac{\mathfrak{E}^s}{\alpha} \cdot \frac{1}{\nu^{i+1}} \cdot \frac{1}{2i+1}.$$

Therefore $\mathfrak{E}^s = (2i+1)\,\alpha\gamma$, and since the α, γ in the case of the P^s, Q^s only differ from these in the accenting of the q's we have

$$\mathfrak{E}^s = (-)^s \frac{i+s!}{i-s!}\left[1 + \beta q_{s-2}\frac{i-s!}{i-s+2!} + \ldots\right]\left[1 + \beta q_{s-2}\frac{i+s-2!}{i+s!} + \ldots\right].$$

E^s = the same with accented q's.

Effecting the multiplication of the series

$$\mathfrak{E}^s = (-)^s \frac{i+s!}{i-s!}\Big[1 + \beta\Big(q_{s-2}\frac{i-s!}{i-s+2!} + q_{s-2}\frac{i+s-2!}{i+s!} + q_{s+2}\frac{i-s!}{i-s-2!} + q_{s+2}\frac{i+s+2!}{i+s!}\Big)$$
$$+ \beta^2\Big(q_{s-4}\frac{i-s!}{i-s+4!} + q_{s-4}\frac{i+s-4!}{i+s!} + q_{s+4}\frac{i-s!}{i-s-4!} + q_{s+4}\frac{i+s+4!}{i+s!} + q_{s-2}q_{s-2}\frac{i-s!}{i+s!}\,\frac{i+s-2!}{i-s+2!}$$
$$+ q_{s+2}q_{s+2}\frac{i-s!}{i+s!}\,\frac{i+s+2!}{i-s-2!} + q_{s-2}q_{s+2}\frac{i-s!}{i+s!}\,\frac{i+s+2!}{i-s+2!} + q_{s+2}q_{s-2}\frac{i-s!}{i+s!}\,\frac{i+s-2!}{i-s-2!}\Big)\Big],$$

E^s = the same with accented q's.

If we substitute for the q's their values, the coefficient of β inside [ ] in the expression for $\mathfrak{E}^s$ is

$$\tfrac{1}{8}\left[-\frac{(i+s)(i+s-1)}{s-1} - \frac{(i-s+1)(i-s+2)}{s-1} + \frac{(i-s)(i-s-1)}{s+1} + \frac{(i+s+1)(i+s+2)}{s+1}\right].$$

In the expression for E^s the first pair of these terms are multiplied by $\frac{s-2}{s}$, and the second pair by $\frac{s+2}{s}$.

The coefficient of β^2 in the expression for $\mathfrak{E}^s$ is

$$\tfrac{1}{128}\Big[\frac{(i+s)(i+s-1)(i+s-2)(i+s-3)}{(s-1)(s-2)}+\frac{(i-s+1)(i-s+2)(i-s+3)(i-s+4)}{(s-1)(s-2)}$$

$$+\frac{(i-s)(i-s-1)(i-s-2)(i-s-3)}{(s+1)(s+2)}+\frac{(i+s+1)(i+s+2)(i+s+3)(i+s+4)}{(s+1)(s+2)}$$

$$+\frac{2(i+s)(i+s-1)(i-s+1)(i-s+2)}{(s-1)^2}+\frac{2(i-s)(i-s-1)(i+s+1)(i+s+2)}{(s+1)^2}$$

$$-\frac{2(i+s-1)(i+s)(i+s+1)(i+s+2)}{(s-1)(s+1)}-\frac{2(i-s-1)(i-s)(i-s+1)(i-s+2)}{(s-1)(s+1)}\Big].$$

In the expression for E^s the first pair of these terms are multiplied by $\frac{s-4}{s}$; the second pair by $\frac{s+4}{s}$; the first of the third pair by $\left(\frac{s-2}{s}\right)^2$, and the second by $\left(\frac{s+2}{s}\right)^2$; and the last pair by $\frac{s^2-4}{s^2}$.

The reduction of terms such as these will occur frequently hereafter, and I will therefore say a word on the most convenient way of carrying it out. It is obvious that the coefficient of β may be arranged in the form

$$Ai(i+1)+B(2i+1)+C.$$

The coefficient A is equal to the coefficient of i^2 in the original expression, and if we put $i=0$ we have $B+C$, and with $i=-1$, $-B+C$. Hence A, B, C may be easily determined.

Again the coefficient of β^2 may be arranged in the form

$$Ai^2(i+1)^2+B(2i+1)i(i+1)+Ci(i+1)+D(2i+1)+E.$$

This may be written

$$Ai^4+2(A+B)i^3+(A+3B+C)i^2+(B+C+D)i+D+E.$$

It is easy to pick out the coefficients of i^4, i^3, i^2, and we thus obtain A, B, C. Then putting i successively equal to 0 and -1 we have $D+E$ and $-D+E$.

In order to express the results succinctly I use as before the notation

$$\Sigma^s=\frac{i(i+1)}{s^2-1},\qquad \Upsilon_i^s=\frac{(i-1)i(i+1)(i+2)}{s^2-4};$$

and I usually omit the superscript and subscript s and i.

Proceeding in this way I find

$$\left.\begin{aligned} \mathfrak{E}_i^s &= (-)^s \frac{i+s\,!}{i-s\,!}\{1 - \tfrac{1}{2}\beta(\Sigma - 1) + \tfrac{1}{32}\beta^2[-s^2(\Sigma^2 + 2\Sigma - 1) \\ &\qquad + 3(\Sigma^2 - 2\Sigma + 2) + 2\Upsilon]\} \\ \mathsf{E}_i^s &= (-)^s \frac{i+s\,!}{i-s\,!}\{1 + \tfrac{1}{2}\beta(\Sigma + 3) + \tfrac{1}{32}\beta^2[s^2(3\Sigma^2 - 2\Sigma + 1) \\ &\qquad - (\Sigma^2 - 26\Sigma - 42) - 2\Upsilon]\} \end{aligned}\right\} \quad . \quad . \ (41).$$

These results may be verified, for if we multiply $\mathfrak{E}^s$ by $\frac{1}{(\mathrm{C}^s)^2}$, as given in (33), we ought to find E^s; and this is so.

The formulæ apparently fail when $s = 0, 1, 2, 3$; but when $s = 3$ they still hold good because, as remarked above, the general formula for $s = 3$ gives correct results when properly interpreted. Thus it only remains to consider $s = 0, 1, 2$.

When $s = 2$ the coefficients of β remain as in (41). In the coefficients of β^2

$$q_{s-4} = 0,\ q_{s-2} = -\tfrac{1}{8}\{i, 2\}\{i, 1\},\ q_{s+2} = \tfrac{1}{8\,.\,3},\ q_{s+4} = \tfrac{1}{128\,.\,3\,.\,4}$$
$$q'_{s-4} = 0,\ q'_{s-2} = 0, \qquad q'_{s+2} = \tfrac{1}{4\,.\,3},\ q'_{s+4} = \tfrac{1}{128\,.\,4}.$$

In the expression for $\mathfrak{E}^2$ the coefficient of β^2 inside the bracket is

$$\begin{aligned} \tfrac{1}{128\,.\,3^2\,.\,4}[&3(i-2)(i-3)(i-4)(i-5) + 3(i+3)(i+4)(i+5)(i+6) \\ &+ 72(i-1)i(i+1)(i+2) + 8(i-2)(i-3)(i+3)(i+4) \\ &- 24(i+1)(i+2)(i+3)(i+4) - 24(i-3)(i-2)(i-1)i]. \end{aligned} \quad (42).$$

Effecting the reduction and writing Σ for $\frac{1}{3}i(i+1)$, we find

$$\mathfrak{E}_i^2 = \frac{i+2\,!}{i-2\,!}\{1 - \tfrac{1}{2}\beta(\Sigma - 1) + \tfrac{1}{256}\beta^2(19\Sigma^2 - 130\Sigma + 80)\} \quad . \quad . \ (43).$$

The coefficient of β^2 for E^2 may be got from (42) thus:—Multiply the first and second terms by 3, erase the third, fifth, and sixth terms, and multiply the fourth term by 4.

Effecting the reduction we find

$$\mathsf{E}_i^2 = \frac{i+2\,!}{i-2\,!}\{1 + \tfrac{1}{2}\beta(\Sigma + 3) + \tfrac{1}{256}\beta^2(25\Sigma^2 + 186\Sigma + 368)\} \quad . \quad . \ (44).$$

Observe that there is no factor by which $\mathfrak{P}^2$ can be converted into P^2, so that this case cannot be verified like the general one.

When $s = 1$ we have

$$q_{s-4} = 0, \quad q_{s-2} = 0, \quad q_{s+2} = \tfrac{1}{16}[1 - \tfrac{1}{16}\beta i(i+1)], \quad q_{s+4} = \tfrac{1}{768}$$
$$q'_{s-4} = 0, \quad q'_{s-2} = 0, \quad q'_{s+2} = \tfrac{3}{16}[1 + \tfrac{1}{16}\beta i(i+1)], \quad q'_{s+4} = \tfrac{5}{768}.$$

The terms in βq_{s+2} and $\beta q'_{s+2}$ now contribute to the terms in β^2.
For $\mathfrak{E}^1$ the term in β inside the bracket is

$$\tfrac{1}{8 \cdot 2}[(i-1)(i-2) + (i+2)(i+3)] = \tfrac{1}{8}[i(i+1) + 4].$$

The term in β^2, of which the first portion is carried over from the term in β, is

$$- \tfrac{1}{256} i(i+1)[(i-1)(i-2) + (i+2)(i+3)]$$
$$+ \tfrac{1}{128 \cdot 2 \cdot 3}[(i-1)(i-2)(i-3)(i-4) + (i+2)(i+3)(i+4)(i+5)$$
$$+ 3(i-1)(i-2)(i+2)(i+3)].$$

This is equal to $- \tfrac{1}{768}[i^2(i+1)^2 - 56i(i+1) - 180]$.

As we cannot now use the abridged notation with Σ_i^1, which is infinite, I write

$$\left.\begin{aligned} j &= i(i+1). \\ \text{Thus} \qquad \mathfrak{E}_i^1 &= - \frac{i+1\,!}{i-1\,!}[1 + \tfrac{1}{8}\beta(j+4) - \tfrac{1}{768}\beta^2(j^2 - 56j - 180)] \end{aligned}\right\} \quad . \quad . \quad (45).$$

For E_i^1 the coefficient of β is three times as great as before, and the coefficient of β^2 is

$$\tfrac{3}{128} i(i+1)[i(i+1) + 4] + \tfrac{1}{128 \cdot 2 \cdot 3}[5(i-1)(i-2)(i-3)(i-4)$$
$$+ 5(i+2)(i+3)(i+4)(i+5) + 27(i-1)(i-2)(i+2)(i+3)].$$

On effecting the reduction I find

$$\mathsf{E}_i^1 = - \frac{i+1\,!}{i-1\,!}[1 + \tfrac{3}{8}\beta(j+4) + \tfrac{1}{768}\beta^2(55j^2 + 376j + 1044)]. \quad . \quad (46).$$

When $s = 0$ we have only $\mathfrak{E}_i$ to determine. Here

$$q_{s-4} = q_{s-2} = 0, \qquad q_{s+2} = \tfrac{1}{4}, \qquad q_{s+4} = \tfrac{1}{128}.$$

The term in β is $\tfrac{1}{4}[i(i-1) + (i+1)(i+2)] = \tfrac{1}{2}(j+1)$.
That in β^2 is

$$\tfrac{1}{128}[i(i-1)(i-2)(i-3) + (i+1)(i+2)(i+3)(i+4)$$
$$+ 8i(i-1)(i+1)(i+2)] = \tfrac{1}{64}(5j^2 + 14j + 12).$$

Therefore

$$\left.\begin{aligned}\mathfrak{E}_i &= 1 + \tfrac{1}{2}\beta(j+1) + \tfrac{1}{64}\beta^2(5j^2 + 14j + 12)\\ &= 1 - \tfrac{1}{2}\beta(\Sigma - 1) + \tfrac{1}{64}\beta^2(5\Sigma^2 - 14\Sigma + 12)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (47).$$

since $\Sigma_i = -i(i+1) = -j.$

Collecting results from (41), (43), (44), (45), 46), and (47),

$$\left.\begin{aligned}
(s > 2)\ \mathfrak{E}_i^s &= (-)^s \frac{i+s\,!}{i-s\,!}\{1 - \tfrac{1}{2}\beta(\Sigma - 1)\\
&\quad + \tfrac{1}{32}\beta^2[-s^2(\Sigma^2 + 2\Sigma - 1) + 3(\Sigma^2 - 2\Sigma + 2) + 2\Upsilon]\},\\
\mathfrak{E}_i^2 &= \frac{i+2\,!}{i-2\,!}\{1 - \tfrac{1}{2}\beta(\Sigma - 1) + \tfrac{1}{256}\beta^2[19\Sigma^2 - 130\Sigma + 80]\},\\
\mathfrak{E}_i^1 &= -\frac{i+1\,!}{i-1\,!}\{1 + \tfrac{1}{8}\beta(j+4) - \tfrac{1}{768}\beta^2(j^2 - 56j - 180]\},\\
\mathfrak{E}_i &= 1 + \tfrac{1}{2}\beta(j+1) + \tfrac{1}{64}\beta^2(5j^2 + 14j + 12).\\
(s > 2)\ \mathsf{E}_i^s &= (-)^s \frac{i+s\,!}{i-s\,!}\{1 + \tfrac{1}{2}\beta(\Sigma + 3)\\
&\quad + \tfrac{1}{32}\beta^2[s^2(3\Sigma^2 - 2\Sigma + 1) - (\Sigma^2 - 26\Sigma - 42) - 2\Upsilon]\},\\
\mathsf{E}_i^2 &= \frac{i+2\,!}{i-2\,!}\{1 + \tfrac{1}{2}\beta(\Sigma + 3) + \tfrac{1}{256}\beta^2[25\Sigma^2 + 186\Sigma + 368]\},\\
\mathsf{E}_i^1 &= -\frac{i+1\,!}{i-1\,!}\{1 + \tfrac{3}{8}\beta(j+4) + \tfrac{1}{768}\beta^2(55j^2 + 376j + 1044)\},\\
\text{where}\quad \Sigma &= \frac{i(i+1)}{s^2-1},\qquad \Upsilon = \frac{(i-1)i(i+1)(i+2)}{s^2-4},\qquad j = i(i+1)
\end{aligned}\right\} (48).$$

PART II.

Application of Ellipsoidal Harmonic Analysis.

§ 11. *The Potential of an harmonic deformation of an Ellipsoid.*

A solid harmonic, or solution of Laplace's equation, is the product of two P-functions of ν and of μ respectively, and of a cosine or sine function of ϕ. A surface harmonic is a P-function of μ multiplied by a cosine or sine function of ϕ.

We found

$$\mathfrak{P}^s(\nu) = \mathrm{P}^s(\nu) + \Sigma\beta^n q_{s-2n}\mathrm{P}^{s-2n}(\nu) + \Sigma\beta^n q_{s+2n}\mathrm{P}^{s+2n}(\nu),$$

where $\mathrm{P}^t(\nu) = \dfrac{(\nu^2-1)^{\frac{1}{2}t}}{2^i\,.\,i\,!}\left(\dfrac{d}{d\nu}\right)^{i+t}(\nu^2-1)^i$; and a similar formula held for $\mathsf{P}^s(\nu)$.

Hitherto we have supposed $\mathrm{P}^t(\mu)$ to have exactly the same form as $\mathrm{P}^t(\nu)$. But since μ is less than unity this introduces an imaginary factor when t is odd, and makes the succession of P's alternately positive and negative when t is even. As this is practically inconvenient I now define

$$P^t(\mu) = \frac{(1 - \mu^2)^{\frac{1}{2}t}}{2^i\, i\,!}\left(\frac{d}{d\mu}\right)^{i+t}(\mu^2 - 1)^i,$$

and then retaining the former meaning for the q coefficients, we give the following definition—

$$\mathfrak{P}^s(\mu) = P^s(\mu) + \Sigma\,(-)^n\beta^n q_{s-2n} P^{s-2n}(\mu) + \Sigma\,(-)^n\beta^n q_{s+2n} P^{s+2n}(\mu)\,;$$

with a similar formula for $\mathsf{P}^s(\mu)$.

Thus we need only remark that in the functions of μ the q's corresponding to odd powers of β enter with the opposite sign from that which holds in the functions of ν, and the whole of our preceding results are true with this definition of $P^t(\mu)$.

If ν_0 defines the ellipsoid to which the surface harmonic applies, we require the expression for the perpendicular p on the tangent plane at ν_0, μ, ϕ, and that for an element of area of the surface of the ellipsoid at the same point.

By the usual formula

$$\frac{k^2}{p^2} = \frac{x^2}{k^2\left(\nu_0^2 - \frac{1+\beta}{1-\beta}\right)^2} + \frac{y^2}{k^2(\nu_0^2 - 1)^2} + \frac{z^2}{k^2\nu_0^2}$$

$$= -\frac{(1-\beta)\left(\mu^2 - \frac{1+\beta}{1-\beta}\right)}{(1+\beta)\left(\nu_0^2 - \frac{1+\beta}{1-\beta}\right)}\cos^2\phi - \frac{\mu^2 - 1}{\nu_0^2 - 1}\sin^2\phi + \frac{\mu^2(1 - \beta\cos 2\phi)}{\nu_0^2(1+\beta)}$$

$$= \frac{(\nu_0^2 - \mu^2)\left(\nu_0^2 - \frac{1-\beta\cos 2\phi}{1-\beta}\right)}{\nu_0^2(\nu_0^2 - 1)\left(\nu_0^2 - \frac{1+\beta}{1-\beta}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49).$$

Let dn, dm, df be the three elements of the orthogonal arcs corresponding to variations of ν, μ, ϕ respectively.

Then by the formula at the end of § 1,

$$\left.\begin{aligned}\left(\frac{dn}{k^2\nu_0 d\nu_0}\right)^2 &= \frac{(\nu_0^2 - \mu^2)\left(\nu_0^2 - \frac{1-\beta\cos 2\phi}{1-\beta}\right)}{k^2\nu_0^2(\nu_0^2 - 1)\left(\nu_0^2 - \frac{1+\beta}{1-\beta}\right)} = \frac{1}{p^2}, \\ \left(\frac{dm}{k^2 d\mu}\right)^2 &= \frac{(\nu_0^2 - \mu^2)\left(\frac{1-\beta\cos 2\phi}{1-\beta} - \mu^2\right)}{k^2(1-\mu^2)\left(\frac{1+\beta}{1-\beta} - \mu^2\right)}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (50).$$

$$\left(\frac{df}{\frac{k\beta\sin 2\phi}{1-\beta}d\phi}\right)^2 = \frac{\left(\nu_0^2 - \frac{1-\beta\cos 2\phi}{1-\beta}\right)\left(\frac{1-\beta\cos 2\phi}{1-\beta} - \mu^2\right)}{\frac{\beta^2\sin^2 2\phi}{(1-\beta)^3}(1 - \beta\cos 2\phi)}.$$

Therefore
$$\left(\frac{df}{kd\phi}\right)^2 = \frac{\left(\nu_0^2 - \frac{1-\beta\cos 2\phi}{1-\beta}\right)\left(\frac{1-\beta\cos 2\phi}{1-\beta} - \mu^2\right)(1-\beta)}{1 - \beta\cos 2\phi} \quad . \quad . \quad . \quad (50),$$

and

$$\left.\begin{aligned}\left(\frac{pdmdf}{k^3d\mu d\phi}\right)^2 &= \nu_0^2(\nu_0^2-1)\left(\nu_0^2-\frac{1+\beta}{1-\beta}\right)(1-\beta)\ \frac{\left(\frac{1-\beta\cos 2\phi}{1-\beta}-\mu^2\right)^2}{(1-\beta\cos 2\phi)(1-\mu^2)\left(\frac{1+\beta}{1-\beta}-\mu^2\right)},\\ \frac{d}{dn} &= \frac{p}{k^2}\cdot\frac{d}{\nu_0 d\nu_0}\end{aligned}\right\} \quad (50).$$

Two functions, written in alternative form,

$$A\mathfrak{P}_i^s\left(\begin{Bmatrix}\nu\\ \nu_0\end{Bmatrix}\right)\mathfrak{Q}_i^s\left(\begin{Bmatrix}\nu_0\\ \nu\end{Bmatrix}\right)\mathfrak{P}_i^s(\mu)\,\mathfrak{C}_i^s(\phi),$$

are solutions of LAPLACE's equation, and together form a function V continuous at the surface of the ellipsoid $\nu=\nu_0$. Reading the upper line we have a function always finite inside the ellipsoid, and reading the lower line one always finite outside. Hence V is the potential of a layer of surface density on the ellipsoid ν_0, and by POISSON's equation that density is equal to $-\frac{1}{4\pi}\left[\frac{dV}{dn}(\text{outside})-\frac{dV}{dn}\ (\text{inside})\right]$.

Let the surface density, which it is our object to find, be

$$p\,\mathfrak{P}_i^s(\mu)\,\mathfrak{C}_i^s(\phi)\,.\,\rho,$$

a surface harmonic multiplied by the perpendicular on to the tangent plane and by a quantity ρ.

Then since $\frac{d}{dn}=\frac{p}{k^2}\ \frac{d}{\nu_0 d\nu_0}$,

$$\rho=-\frac{A}{4\pi k^2\nu_0}\left[\mathfrak{P}_i^s(\nu_0)\frac{d}{d\nu_0}\mathfrak{Q}_i^s(\nu_0)-\mathfrak{Q}_i^s(\nu_0)\frac{d}{d\nu_0}\mathfrak{P}_i^s(\nu_0)\right].$$

But
$$\mathfrak{Q}_i^s(\nu_0)=\mathfrak{E}_i^s\,\mathfrak{P}_i^s(\nu_0)\int_{\nu_0}^{\infty}\frac{d\nu}{[\mathfrak{P}_i^s(\nu)]^2(\nu^2-1)^{\frac{1}{2}}\left(\nu^2-\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}}.$$

Differentiating this logarithmically we find

$$\rho=\frac{A\mathfrak{E}_i^s}{4\pi k^2\nu_0(\nu_0^2-1)^{\frac{1}{2}}\left(\nu_0^2-\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}},\text{ a constant.}$$

Hence surface density $p\,\mathfrak{P}_i^s(\mu)\mathfrak{C}_i^s(\phi)\,.\,\rho$, where ρ is constant, gives rise to potential

$$\begin{Bmatrix}\text{inside}\\ \text{outside}\end{Bmatrix}\text{ the ellipsoid }\ \frac{4\pi k^2\rho\nu_0}{\mathfrak{E}_i^s}(\nu_0^2-1)^{\frac{1}{2}}\left(\nu_0^2-\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}\mathfrak{P}_i^s\left(\begin{Bmatrix}\nu\\ \nu_0\end{Bmatrix}\right)\mathfrak{Q}_i^s\left(\begin{Bmatrix}\nu_0\\ \nu\end{Bmatrix}\right)\mathfrak{P}_i^s(\mu)\,\mathfrak{C}_i^s(\phi) \quad (51).$$

The same investigation holds good with $\mathfrak{S}_i^s(\phi)$, or with **P**, **Q**, **C**, **S** in place of the corresponding letters above.

Imagine that the surface of a homogeneous ellipsoid of density ρ, defined by ν_0, receives a normal displacement δn, such that

$$\delta n = p \cdot \epsilon \mathfrak{P}_i^s(\mu) \mathfrak{C}_i^s(\phi).$$

Then the equivalent surface density is $p \cdot \epsilon\rho \mathfrak{P}_i^s(\mu)\mathfrak{C}_i^s(\phi)$, and we can at once write down the expressions for the internal and external potentials by means of (51).

If x_0, y_0, z_0 be the co-ordinates of a point on the surface, it is clear that the co-ordinates of the corresponding point on the deformed surface are

$$x = x_0\left(1 + \frac{p^2\epsilon\mathfrak{P}_i^s(\mu)\mathfrak{C}_i^s(\phi)}{k^2\left(\nu_0^2 - \frac{1+\beta}{1-\beta}\right)}\right), \quad y = y_0\left(1 + \frac{p^2\epsilon\mathfrak{P}_i^s(\mu)\mathfrak{C}_i^s(\phi)}{k^2(\nu_0^2 - 1)}\right),$$

$$z = z_0\left(1 + \frac{p^2\epsilon\mathfrak{P}_i^s(\mu)\mathfrak{C}_i^s(\phi)}{k^2\nu^2}\right).$$

Hence the equation to the deformed surface is

$$\frac{x^2}{k^2\left(\nu_0^2 - \frac{1+\beta}{1-\beta}\right)} + \frac{y^2}{k^2(\nu_0^2 - 1)} + \frac{z^2}{k^2\nu_0^2} = 1 + 2\epsilon\mathfrak{P}_i^s(\mu)\mathfrak{C}_i^s(\phi) \quad . \quad . \qquad (52),$$

or since $\dfrac{x^2}{k^2\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)} + \dfrac{y^2}{k^2(\nu^2-1)} + \dfrac{z^2}{k^2\nu^2} = 1$, it may be written

$$(\nu^2 - \nu_0^2)\frac{k^2}{p^2} = 2\epsilon\mathfrak{P}_i^s(\mu)\mathfrak{C}_i^s(\phi).$$

If we substitute for $\dfrac{k^2}{p^2}$ its value from (49), this may be written in the form

$$\frac{(\nu^2 - \nu_0^2)(\mu^2 - \nu_0^2)\left(\frac{1-\beta\cos 2\phi}{1-\beta} - \nu_0^2\right)}{\nu_0^2(\nu_0^2 - 1)\left(\nu_0^2 - \frac{1+\beta}{1-\beta}\right)} = 2\epsilon\mathfrak{P}_i^s(\mu)\mathfrak{C}_i^s(\phi) \quad . \quad . \quad . \quad . \qquad (52).$$

This is the equation in elliptic co-ordinates to the deformed surface, but in actual computation the form involving rectangular co-ordinates might perhaps be more convenient.

§ 12. *The Potential of a homogeneous solid Ellipsoid.*

It is well known that the potential of a solid ellipsoid externally is equal to that of a "focaloid" shell of the same mass coincident with its external surface.

If ρ' be the density of the shell defined by ν_0 and $\nu_0 + \delta\nu$, we have

$$\tfrac{4}{3}\pi k^3\rho'\left[\left(\nu_0^2 + 2\nu_0\delta\nu - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}(\nu_0^2 + 2\nu_0\delta\nu - 1)^{\frac{1}{2}}(\nu_0^2 + 2\nu_0\delta\nu)^{\frac{1}{2}}\right.$$

$$\left. - \left(\nu_0^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}(\nu_0^2 - 1)^{\frac{1}{2}}\nu_0\right] = \tfrac{4}{3}\pi k^3\rho\left(\nu_0^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}(\nu_0^2 - 1)^{\frac{1}{2}}\nu_0.$$

Therefore $$\rho' \nu_0 \delta\nu \left(\frac{1}{\nu_0^2 - \frac{1+\beta}{1-\beta}} + \frac{1}{\nu_0^2 - 1} + \frac{1}{\nu_0^2} \right) = \rho,$$

or $$\rho' \nu_0 \delta\nu = \frac{\rho \nu_0^2 (\nu_0^2 - 1)\left(\nu_0^2 - \frac{1+\beta}{1-\beta}\right)}{3\nu_0^4 - \frac{4\nu_0^2}{1-\beta} + \frac{1+\beta}{1-\beta}} \quad \ldots \ldots \ldots \quad (53).$$

If δn be the thickness of the shell at the point where p is the perpendicular on the tangent plane,

$$\delta n = \nu_0 \delta\nu \cdot \frac{k^2}{p}.$$

If we multiply both sides of (53) by $\frac{k^2}{p}$, we see that the surface density of the focaloid shell is

$$p \cdot \frac{\rho \nu_0^2 (\nu_0^2 - 1)\left(\nu_0^2 - \frac{1+\beta}{1-\beta}\right)}{3\nu_0^4 - \frac{4\nu_0^2}{1-\beta} + \frac{1+\beta}{1-\beta}} \cdot \frac{k^2}{p^2}.$$

If therefore we can express $\frac{k^2}{p^2}$ in the form of surface harmonics, it will be easy to write down the external potential of the ellipsoid by means of the formula (51).

Before doing this I will, however, take one other step.

It is easy to see that

$$3\nu_0^4 - \frac{4\nu_0^2}{1-\beta} + \frac{1+\beta}{1-\beta} = 3\left(\nu_0^2 - \frac{2+B}{3(1-\beta)}\right)\left(\nu_0^2 - \frac{2-B}{3(1-\beta)}\right) \quad . \quad . \quad . \quad (54),$$

where for brevity $B = (1 + 3\beta^2)^{\frac{1}{2}}$.

Now on referring to § 7, (17) and (23), we see that

$$\mathfrak{P}_2(\nu) = \mathrm{P}_2(\nu) + \frac{B-1}{6\beta} \mathrm{P}_2^2(\nu), \qquad \mathfrak{P}_2^2(\nu) = -\frac{2(B-1)}{\beta} \mathrm{P}_2(\nu) + \mathrm{P}_2^2(\nu),$$

where $$\mathrm{P}_2(\nu) = \tfrac{3}{2}\nu^2 - \tfrac{1}{2}, \qquad \mathrm{P}_2^2(\nu) = 3(\nu^2 - 1).$$

If then we put $$\mathfrak{P}_2(\nu) = \alpha\nu^2 + \gamma,$$

$$\mathfrak{P}_2^2(\nu) = -\alpha'\nu^2 - \gamma', \qquad \text{or} \qquad \mathfrak{P}_2^2(\mu) = \alpha'\mu^2 + \gamma',$$

it is clear that

$$\alpha = \frac{B - 1 + 3\beta}{2\beta}, \qquad \gamma = \frac{-B + 1 - \beta}{2\beta};$$

$$\alpha' = \frac{3(B - 1 - \beta)}{\beta}, \qquad \gamma' = \frac{-B + 1 + 3\beta}{\beta};$$

and $$\frac{\gamma}{\alpha} = \frac{B-2}{3(1-\beta)}, \qquad \frac{\gamma'}{\alpha'} = \frac{-B-2}{3(1-\beta)}.$$

It is obvious then that our expression (54) is equal to $-\frac{3}{\alpha\alpha'}\,\mathfrak{P}_2(\nu_0)\,\mathfrak{P}_2^2(\nu_0)$. Then since $-\frac{3}{\alpha\alpha'} = \frac{1+B}{3(1-\beta)}$, we have the surface density of the focaloid given by

$$p \cdot \frac{3(1+\beta)}{1+B} \cdot \frac{\rho\nu_0^2(\nu_0^2-1)\left(\nu_0^2-\frac{1+\beta}{1-\beta}\right)}{\mathfrak{P}_2(\nu_0)\,\mathfrak{P}_2^2(\nu_0)} \cdot \left(\frac{k}{p}\right)^2,$$

where
$$\frac{k^2}{p^2} = \frac{x_0^2}{k^2\left(\nu_0^2-\frac{1+\beta}{1-\beta}\right)^2} + \frac{y_0^2}{k^2(\nu_0^2-1)^2} + \frac{z_0^2}{k^2\nu_0^4}.$$

But since
$$\frac{x_0^2}{k^2\left(\nu_0^2-\frac{1+\beta}{1-\beta}\right)} + \frac{y_0^2}{k^2(\nu_0^2-1)} + \frac{z_0^2}{k^2\nu_0^2} = 1 \quad \ldots \ldots \quad (55),$$

we have
$$\frac{k^2}{p^2} = \frac{1}{\nu_0^2} + \frac{\frac{1+\beta}{1-\beta}\cdot x_0^2}{k^2\nu_0^2\left(\nu_0^2-\frac{1+\beta}{1-\beta}\right)^2} + \frac{y_0^2}{k^2\nu_0^2(\nu_0^2-1)^2} \quad \ldots \ldots \quad (56).$$

With the object of writing this function in surface harmonics, and besides to enable us to express a rotation potential in similar form, we have to reduce x^2, y^2, z^2 in the required manner.

I now drop the suffix zero, since we are not concerned with any particular ellipsoid.

Referring again to § 7, (18) and (24), we have

$$\mathfrak{C}_2(\phi) = 1 - \frac{B-1}{\beta}\cos 2\phi, \qquad \mathfrak{C}_2^2(\phi) = \frac{B-1}{3\beta} + \cos 2\phi.$$

If then we put

$$\epsilon = \frac{2(1-B)}{\beta}, \qquad \zeta = \frac{B-1+\beta}{\beta};$$

$$\epsilon' = 2, \qquad \zeta' = \frac{B-1-3\beta}{3\beta};$$

we may write

$$\mathfrak{C}_2(\phi) = \epsilon\cos^2\phi + \zeta, \qquad \mathfrak{C}_2^2(\phi) = \epsilon'\cos^2\phi + \zeta'.$$

Let us assume, if possible,

$$\frac{-x^2}{k^2\left(\frac{1-\beta}{1+\beta}\right)\left(\nu^2-\frac{1+\beta}{1-\beta}\right)} = F\,\mathfrak{P}_2(\mu)\,\mathfrak{C}_2(\phi) + G\,\mathfrak{P}_2^2(\mu)\,\mathfrak{C}_2^2(\phi) + H,$$

or
$$\left(\mu^2 - \frac{1+\beta}{1-\beta}\right)\cos^2\phi = F(\alpha\mu^2+\gamma)(\epsilon\cos^2\phi+\zeta) + G(\alpha'\mu^2+\gamma')(\epsilon'\cos^2\phi+\zeta') + H.$$

From which it follows that

$$F\alpha\zeta + G\alpha'\zeta' = 0, \qquad F\gamma\zeta + G\gamma'\zeta' + H = 0,$$

$$F\alpha\epsilon + G\alpha'\epsilon' = 1, \qquad F\gamma\epsilon + G\gamma'\epsilon' = -\frac{1+\beta}{1-\beta}.$$

These equations give

$$F = \frac{1}{\alpha\zeta}\cdot\frac{1}{\frac{\epsilon}{\zeta}-\frac{\epsilon'}{\zeta'}},\qquad G = -\frac{1}{\alpha'\zeta'}\cdot\frac{1}{\frac{\epsilon}{\zeta}-\frac{\epsilon'}{\zeta'}},\qquad H = -\frac{\frac{\gamma}{\alpha}-\frac{\gamma'}{\alpha'}}{\frac{\epsilon}{\zeta}-\frac{\epsilon'}{\zeta'}},$$

and the condition

$$(1-\beta)\left(\frac{\gamma\epsilon}{\alpha\zeta}-\frac{\gamma'\epsilon'}{\alpha'\zeta'}\right)+(1+\beta)\left(\frac{\epsilon}{\zeta}-\frac{\epsilon'}{\zeta'}\right)=0.$$

Now

$$\frac{\gamma\epsilon}{\alpha\zeta}=\frac{1+\beta-B}{1-\beta},\qquad \frac{\gamma'\epsilon'}{\alpha'\zeta'}=\frac{1+\beta+B}{1-\beta},$$

$$\frac{\epsilon}{\zeta}=\frac{-1-3\beta+B}{1+\beta},\qquad \frac{\epsilon'}{\zeta'}=\frac{-1-3\beta-B}{1+\beta}.$$

Since these values satisfy the condition amongst the coefficients, the assumed form for x^2 is justifiable.

I find then

$$F=\frac{1+B}{2B}\cdot\frac{B-2\beta}{3(1-\beta)},\qquad G=-\frac{1+B}{4B}\,\frac{B+2\beta}{3(1-\beta)},\qquad H=-\frac{1+\beta}{3(1-\beta)}.$$

Whence

$$\frac{3x^2}{k^2\left(\nu^2-\frac{1+\beta}{1-\beta}\right)}=-\frac{1+B}{2B}\cdot\frac{B-2\beta}{1+\beta}\,\mathfrak{P}_2(\mu)\mathfrak{C}_2(\phi)+\frac{1+B}{4B}\cdot\frac{B+2\beta}{1+\beta}\,\mathfrak{P}_2^2(\mu)\mathfrak{C}_2^2(\phi)+1\,. \quad (57).$$

This is the required expression for x^2 in surface harmonics.

Next assume

$$\frac{-y^2}{k^2(\nu^2-1)}=F_1\mathfrak{P}_2(\mu)\mathfrak{C}_2(\phi)+G_1\mathfrak{P}_2^2(\mu)\mathfrak{C}_2^2(\phi)+H_1.$$

If we put

$$\mathfrak{C}_2(\phi)=\epsilon_1\sin^2\phi+\zeta_1,\qquad \mathfrak{C}_2^2(\phi)=\epsilon_1'\sin^2\phi+\zeta_1';$$

we have

$$\epsilon_1=\frac{2(B-1)}{\beta},\qquad \zeta_1=\frac{-B+1+\beta}{\beta};$$

$$\epsilon_1'=-2,\qquad \zeta_1'=\frac{B-1+3\beta}{3\beta};$$

and

$$(\mu^2-1)\sin^2\phi=F_1(\alpha\mu^2+\gamma)(\epsilon_1\sin^2\phi+\zeta_1)+G_1(\alpha'\mu^2+\zeta')(\epsilon_1'\sin^2\phi+\zeta_1')+H_1.$$

Whence F_1, G_1, H_1 have the same forms as before, and the condition to be satisfied by the coefficients is

$$\frac{\gamma\epsilon_1}{\alpha\zeta_1}-\frac{\gamma'\epsilon_1'}{\alpha'\zeta_1'}+\frac{\epsilon_1}{\zeta_1}-\frac{\epsilon_1'}{\zeta_1'}=0.$$

It will be found that the condition is satisfied, and that

$$F_1=\frac{1+B}{6B},\qquad G_1=\frac{1+B}{12B},\qquad H_1=-\tfrac{1}{3}.$$

Hence

$$\frac{3y^2}{k^2(\nu^2-1)} = -\frac{1+B}{2B}\mathfrak{P}_2(\mu)\mathfrak{C}_2(\phi) - \frac{1+B}{4B}\mathfrak{P}_2^{\,2}(\mu)\mathfrak{C}_2^{\,2}(\phi) + 1 \quad . \quad . \quad (58).$$

It follows from (55) that

$$\frac{3z^2}{k^2\nu^2} = \frac{1+B}{2B}\left(\frac{B-2\beta}{1+\beta}+1\right)\mathfrak{P}_2(\mu)\mathfrak{C}_2(\phi) - \frac{1+B}{4B}\left(\frac{B+2\beta}{1+\beta}-1\right)\mathfrak{P}_2^{\,2}(\mu)\mathfrak{C}_2^{\,2}(\phi) + 1 \quad . \quad . \quad . \quad . \quad (59).$$

Whence

$$\frac{3}{k^2}(y^2+z^2) = \frac{1+B}{2B}\left(1+\nu^2\frac{B-2\beta}{1+\beta}\right)\mathfrak{P}_2(\mu)\mathfrak{C}_2(\phi)$$
$$+\frac{1+B}{4B}\left(1-\nu^2\frac{B+2\beta}{1+\beta}\right)\mathfrak{P}_2(\mu)\mathfrak{C}_2(\phi) + 2\nu^2 - 1 \quad . \quad . (60).$$

This is needed to express the rotation potential $\frac{1}{2}\omega^2(y^2+z^2)$. If we add $\frac{3x^2}{k^2}$ to this we have

$$\frac{3}{k^2}(x^2+y^2+z^2) = \frac{1+B}{2B}\cdot\frac{B+1-3\beta}{1-\beta}\mathfrak{P}_2(\mu)\mathfrak{C}_2(\phi)$$
$$-\frac{1+B}{4B}\cdot\frac{B-1+3\beta}{1-\beta}\mathfrak{P}_2^{\,2}(\mu)\mathfrak{C}_2^{\,2}(\phi) + 3\nu^2 - \frac{2}{1-\beta} \quad . \quad . (61).$$

This expression will be needed hereafter.

Returning now to the formation of the expression for k^2/p^2, I find

$$\frac{3k^2}{p^2} = \frac{1}{\nu^2(\nu^2-1)\left(\nu^2-\frac{1+\beta}{1-\beta}\right)}\Bigg[-\frac{1+B}{2B}\cdot\frac{\nu^2(B+1-3\beta)-(B+1-\beta)}{1-\beta}\mathfrak{P}_2(\mu)\mathfrak{C}_2(\phi)$$
$$+\frac{1+B}{4B}\cdot\frac{\nu^2(B-1+3\beta)-(B-1+\beta)}{1-\beta}\mathfrak{P}_2^{\,2}(\mu)\mathfrak{C}_2^{\,2}(\phi)$$
$$+3\nu^4 - \frac{4\nu^2}{1-\beta} + \frac{1+\beta}{1-\beta}\Bigg].$$

On considering the forms of the functions $\mathfrak{P}_2(\nu)$, $\mathfrak{P}_2^{\,2}(\nu)$, it is found that this result may be written thus:

$$\frac{k^2}{p^2} = \frac{\mathfrak{P}_2(\nu)\mathfrak{P}_2^{\,2}(\nu)}{3\nu^2(\nu^2-1)\left(\nu^2-\frac{1+\beta}{1-\beta}\right)}\cdot\frac{1+B}{3(1-\beta)}\left[\frac{1+B}{2B}\frac{\mathfrak{P}_2(\mu)\mathfrak{C}_2(\phi)}{\mathfrak{P}_2(\nu)} + \frac{3\beta}{2B}\frac{\mathfrak{P}_2^{\,2}(\mu)\mathfrak{C}_2^{\,2}(\phi)}{\mathfrak{P}_2^{\,2}(\nu)} + 1\right].$$

Therefore, writing $\mathfrak{P}_0(\mu)\mathfrak{C}_0(\phi)$ for unity, the surface density of the focaloid shell, for which $\nu = \nu_0$, is

$$p\rho\left[\frac{1+B}{6B}\frac{\mathfrak{P}_2(\mu)\mathfrak{C}_2(\phi)}{\mathfrak{P}_2(\nu_0)} + \frac{\beta}{2B}\frac{\mathfrak{P}_2^{\,2}(\mu)\mathfrak{C}_2^{\,2}(\phi)}{\mathfrak{P}_2^{\,2}(\nu_0)} + \tfrac{1}{3}\mathfrak{P}_0(\mu)\mathfrak{C}_0(\phi)\right] \quad . \quad . \quad . \quad .(62)$$

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By means of (51), we now at once write down the external potential of the ellipsoid. It is

$$V = \frac{M_0}{k}\left[\frac{1+B}{2B}\,\frac{\mathfrak{Q}_2(\nu)}{\mathfrak{E}_2}\,\mathfrak{P}_2(\mu)\mathfrak{C}_2(\phi) + \frac{3\beta}{2B}\,\frac{\mathfrak{Q}_2^2(\nu)}{\mathfrak{E}_2^2}\,\mathfrak{P}_2^2(\mu)\mathfrak{C}_2^2(\phi) + \frac{\mathfrak{Q}_0(\nu)}{\mathfrak{E}_0}\right] \quad . \; (63).$$

In this expression M_0 denotes the mass of the ellipsoid, and the $\mathfrak{E}$'s are merely coefficients determined approximately in § 10.

In order to find the potential internally, let

$$r^2 = x^2 + y^2 + z^2;$$

and, as suggested by the form of (61), let

$$\frac{r_0^2}{k^2} = \frac{1+B}{6B}\cdot\frac{B+1-3\beta}{1-\beta}\,\frac{\mathfrak{P}_2(\nu)}{\mathfrak{P}_2(\nu_0)}\,\mathfrak{P}_2(\mu)\mathfrak{C}_2(\phi)$$
$$- \frac{1+B}{12B}\cdot\frac{B-1+3\beta}{1-\beta}\,\frac{\mathfrak{P}_2^2(\nu)}{\mathfrak{P}_2^2(\nu)_0}\,\mathfrak{P}_2^2(\mu)\mathfrak{C}_2^2(\phi) + \nu_0^2 - \frac{2}{3(1-\beta)}.$$

Then r_0^2 is a solution of LAPLACE's equation throughout the interior of the ellipsoid, and at the surface, where $\nu = \nu_0$, it is equal to $x^2 + y^2 + z^2$.

Now consider the function

$$V = -\tfrac{2}{3}\pi\rho\,(r^2 - r_0^2) + \frac{M_0}{k}\left[\frac{1+B}{2B}\,\frac{\mathfrak{Q}_2(\nu_0)}{\mathfrak{E}_2\mathfrak{P}_2(\nu_0)}\,\mathfrak{P}_2(\nu)\mathfrak{P}_2(\mu)\mathfrak{C}_2(\phi)\right.$$
$$\left. + \frac{3\beta}{2B}\,\frac{\mathfrak{Q}_2^2(\nu_0)}{\mathfrak{E}_2^2\mathfrak{P}_2^2(\nu_0)}\,\mathfrak{P}_2^2(\nu)\mathfrak{P}_2^2(\mu)\mathfrak{C}_2^2(\phi) + \frac{\mathfrak{Q}_0(\nu_0)}{\mathfrak{E}_0}\right] . \quad . \; (64).$$

The whole of it, excepting the term in r^2, is a solution of LAPLACE's equation for space inside the ellipsoid, and the term in r^2 gives $\nabla^2 V = -4\pi\rho$. Also at the surface, where $\nu = \nu_0$, this expression agrees with (63). Hence we have found the potential of the ellipsoid internally.

The potential at an internal point does not lend itself to expression in elliptic co-ordinates, but it may be given another form which is perhaps more convenient.

In our present notation the well-known formula is

$$V = \frac{3}{2}\frac{M_0}{k}\int_{\nu_0}^{\infty}\left(1 - \frac{x^2}{k^2\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)} - \frac{y^2}{k^2(\nu^2-1)} - \frac{z^2}{k^2\nu^2}\right)\frac{d\nu}{\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}(\nu^2-1)^{\frac{1}{2}}}.$$

Since $\mathfrak{P}_0(\nu) = 1$, $\mathsf{P}_1^1(\nu) = \left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}$, $\mathfrak{P}_1^1(\nu) = (\nu^2-1)^{\frac{1}{2}}$, $\mathfrak{P}_1(\nu) = \nu$, the integrals may be expressed in terms of the Q-functions, and we have (omitting the divisors $\mathfrak{E}$ and E for brevity)

$$V = \frac{3}{2}\frac{M_0}{k}\left(\frac{\mathfrak{Q}_0(\nu_0)}{\mathfrak{P}_0(\nu_0)} - \frac{x^2}{k^2}\frac{\mathsf{Q}_1^1(\nu_0)}{\mathsf{P}_1^1(\nu_0)} - \frac{y^2}{k^2}\frac{\mathfrak{Q}_1^1(\nu_0)}{\mathfrak{P}_1^1(\nu_0)} - \frac{z^2}{k^2}\frac{\mathfrak{Q}_1(\nu_0)}{\mathfrak{P}_1(\nu_0)}\right) \quad . \quad . \quad . \; (65).$$

In this we may substitute the expressions for x^2, y^2, z^2 found above.

It may be worth noting that

$$\frac{\mathsf{Q}_1^{\,1}(\nu)}{\mathsf{P}_1^{\,1}(\nu)} + \frac{\mathfrak{Q}_1^{\,1}(\nu)}{\mathfrak{P}_1^{\,1}(\nu)} + \frac{\mathfrak{Q}_1(\nu)}{\mathfrak{P}_1(\nu)} = \frac{1}{\nu(\nu^2-1)^{\frac{1}{2}}\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}}.$$

Also $\quad \mathsf{P}_1^{\,1}(\nu)\,\mathsf{Q}_1^{\,1}(\nu) + \mathfrak{P}_1^{\,1}(\nu)\,\mathfrak{Q}_1^{\,1}(\nu) + \mathfrak{P}_1(\nu)\,\mathfrak{Q}_1(\nu) + \mathfrak{P}_0(\nu)\,\mathfrak{Q}_0(\nu) = 0\,.$

This last follows from the fact that if a, b, c are the axes of the ellipsoid, and if Ψ denotes the function $\int_0^\infty \frac{du}{\text{ABC}}$ (proportional to our $\mathfrak{Q}_0(\nu_0)$), Ψ is a homogeneous function of degree -1 in a, b, c, and therefore

$$a\frac{d\Psi}{da} + b\frac{d\Psi}{db} + c\frac{d\Psi}{dc} = -\Psi\,.$$

§ 13. *Preparation for the Integration of the square of a surface harmonic over the Ellipsoid.*

If it is intended to express any function in harmonics, it is necessary to know the integrals over the surface of the ellipsoid of the squares of surface harmonics multiplied by the perpendicular on the tangent plane.

The surface harmonic has one of the eight forms

$$\text{V}_i^s = [\mathfrak{P}_i^s(\mu) \quad \text{or} \quad \mathsf{P}_i^s(\mu)] \times \begin{bmatrix} \mathfrak{C}_i^s(\phi) & \text{or} & \mathsf{C}_i^s(\phi) \\ \mathfrak{S}_i^s(\phi) & & \mathsf{S}_i^s(\phi) \end{bmatrix};$$

and the P-functions are expressible in terms of the P's where

$$\text{P}_i^s(\mu) = \frac{(1-\mu^2)^{\frac{1}{2}s}}{2^i\,.\,i\,!}\left(\frac{d}{d\mu}\right)^{i+s}(\mu^2-1)^i\,.$$

I shall in this portion of the investigation frequently write $\mu = \sin\theta$, and shall omit the μ or θ or ϕ in the P-, C-, S-functions. Also I may very generally omit the subscript i, as elsewhere.

If $d\sigma$ denotes the element of surface of the ellipsoid, and

$$\text{M} = k^3\nu(\nu^2-1)^{\frac{1}{2}}\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}},$$

so that $\frac{4}{3}\pi\,\text{M}$ is the volume of the ellipsoid, we have, by (50) of § 11,

$$\frac{p\,d\sigma}{d\theta d\phi} = \frac{\text{M}(1-\beta)^{\frac{1}{2}}}{(1-\beta\cos 2\phi)^{\frac{1}{2}}}\,\frac{\left(\cos^2\theta + \frac{\beta}{1-\beta} - \frac{\beta\cos 2\phi}{1-\beta}\right)}{\left(\frac{1+\beta}{1-\beta} - \sin^2\theta\right)^{\frac{1}{2}}}\,.$$

Then $$\int p(\text{V}_i^s)^2 d\sigma = \text{M}(1-\beta)^{\frac{1}{2}}\iint \frac{\cos^2\theta + \frac{\beta}{1-\beta} - \frac{\beta\cos 2\phi}{1-\beta}}{(1-\beta\cos 2\phi)^{\frac{1}{2}}\left(\frac{1+\beta}{1-\beta} - \sin^2\theta\right)^{\frac{1}{2}}}(\text{V}_i^s)^2 d\theta d\phi,$$

where the limits of θ are $\frac{1}{2}\pi$ to $-\frac{1}{2}\pi$, and of ϕ are 2π to 0.

It will be legitimate to develop $pd\sigma$ in powers of $\sec^2\theta$ up to any given power, provided $(\mathrm{V}_i^s)^2$ involves as a factor such a power of $\cos^2\theta$ that the whole function to be integrated does not become infinite at the poles where $\theta = \pm\frac{1}{2}\pi$.

I shall at present limit the developments to the square of β.

We know that P^s is of the same form as $\mathfrak{P}^s$, but with the additional factor

$$\left(\frac{\frac{1+\beta}{1-\beta} - \sin^2\theta}{\cos^2\theta}\right)^{\frac{1}{2}}.$$

Suppose then that

$$\Pi_0 + \beta\Pi_1 + \beta^2\Pi_2 = (\mathfrak{P}^s)^2 \quad \text{or} \quad \frac{\cos^2\theta}{\frac{1+\beta}{1-\beta} - \sin^2\theta}(\mathsf{P}^s)^2\,;$$

and let
$$\gamma = 1 - \cos 2\phi.$$

Then we put
$$F_1 = \frac{\cos^2\theta + \gamma(\beta+\beta^2)}{\left(\frac{1+\beta}{1-\beta} - \sin^2\theta\right)^{\frac{1}{2}}}(\Pi_0 + \beta\Pi_1 + \beta^2\Pi_2),$$

and
$$F_2 = \frac{\frac{1+\beta}{1-\beta} - \sin^2\theta}{\cos^2\theta}F_1.$$

Now suppose that K^2, a function independent of θ, denotes one of the four

$$\frac{(\mathfrak{C}^s)^2 \text{ or } (\mathfrak{S}^s)^2 \text{ or } (\mathsf{C}^s)^2 \text{ or } (\mathsf{S}^s)^2}{(1-\beta\cos 2\phi)^{\frac{1}{2}}}.$$

Then in the cases involving $\mathfrak{P}$-functions and P-functions respectively, we have in alternative form—

$$\text{for } \begin{Bmatrix}\mathfrak{P}^s\\ \mathsf{P}^s\end{Bmatrix}, \quad \int p(\mathrm{V}_i^s)^2 d\sigma = \mathrm{M}(1-\beta)^{\frac{1}{2}}\iint K^2 \begin{Bmatrix}F_1\\ F_2\end{Bmatrix} d\theta d\phi.$$

If it be supposed that the development in powers of $\sec^2\theta$ is justifiable

$$F_1 = \cos\theta\left[1 + \frac{\gamma(\beta+\beta^2)}{\cos^2\theta}\right]\left[1 - \frac{\beta+\beta^2}{\cos^2\theta} + \frac{3}{2}\frac{\beta^2}{\cos^4\theta}\right][\Pi_0 + \beta\Pi_1 + \beta^2\Pi_2],$$

$$= \Pi_0\cos\theta + \beta\left[\frac{\Pi_0(\gamma-1)}{\cos\theta} + \Pi_1\cos\theta\right] + \beta^2\left[\frac{\Pi_0(\gamma-1)}{\cos\theta} + \frac{\Pi_0(\frac{3}{2}-\gamma)}{\cos^3\theta} + \frac{\Pi_1(\gamma-1)}{\cos\theta} + \Pi_2\cos\theta\right].$$

And F_2 has a similar form, save that $\gamma + 1$ replaces $\gamma - 1$, and $\gamma - \frac{1}{2}$ replaces $\frac{3}{2} - \gamma$.

It is clear that unless Π_0 is divisible by $\cos^3\theta$ and Π_1 by $\cos\theta$, $\int F_1 d\theta$ and $\int F_2 d\theta$ will have infinite elements at the poles, and the development is not legitimate.

Since $\mathrm{P}^s = \frac{\cos^s\theta}{2^i . i!}\frac{d^{i+s}}{d\mu}(\mu^2-1)^i$, it follows that the power of $\cos\theta$ by which P^s is divisible increases as s increases.

Let us consider the case of $s = 2$.

Then

$$\Pi_0 + \beta\Pi_1 + \beta^2\Pi_2 = [\mathrm{P}^2 - \beta q_0 \mathrm{P} - \beta q_4 \mathrm{P}^4 + \beta^2 q_6 \mathrm{P}^6]^2$$
$$= (\mathrm{P}^2)^2 - 2\beta(q_0 \mathrm{P}\mathrm{P}^2 + q_4 \mathrm{P}^2\mathrm{P}^4)$$
$$+ \beta^2[2q_6\mathrm{P}^2\mathrm{P}^6 + (q_0\mathrm{P})^2 + (q_4\mathrm{P}^4)^2 + 2q_0q_4\mathrm{P}\mathrm{P}^4]$$

(or the same with accented q's for the other case).

From this it is clear that Π_0 is divisible by $\cos^4\theta$ and Π_1 by $\cos^2\theta$, and the method of development is legitimate when $s = 2$, but it is not so when $s = 0$ and $s = 1$.

The investigation then separates into the general case, and the cases $s = 0$, $s = 1$.

§ 14. *Integration in the general case.*

We have

$$\mathfrak{P}^s = \mathrm{P}^s - \beta q_{s-2}\mathrm{P}^{s-2} - \beta q_{s+2}\mathrm{P}^{s+2} + \beta^2 q_{s-4}\mathrm{P}^{s-4} + \beta^2 q_{s+4}\mathrm{P}^{s+4},$$

and

$$(\mathfrak{P}^s)^2 = (\mathrm{P}^s)^2 - 2\beta(q_{s-2}\mathrm{P}^s\mathrm{P}^{s-2} + q_{s+2}\mathrm{P}^s\mathrm{P}^{s+2}) + 2\beta^2(q_{s-4}\mathrm{P}^s\mathrm{P}^{s-4} + q_{s+4}\mathrm{P}^s\mathrm{P}^{s+4})$$
$$+ \beta^2[(q_{s-2}\mathrm{P}^{s-2})^2 + (q_{s+2}\mathrm{P}^{s+2})^2 + 2q_{s-2}q_{s+2}\mathrm{P}^{s-2}\mathrm{P}^{s+2}].$$

Also $\left[\mathsf{P}^s\left(\dfrac{\cos^2\theta}{\frac{1+\beta}{1-\beta} - \sin^2\theta}\right)^{\frac{1}{2}}\right]^2$ has the same form with accented q's, so that it will be merely necessary to accent the q's to obtain the second case.

We have then

$$\Pi_0 = (\mathrm{P}^s)^2, \quad \Pi_1 = -2(q_{s-2}\mathrm{P}^s\mathrm{P}^{s-2} + q_{s+2}\mathrm{P}^s\mathrm{P}^{s+2}),$$
$$\Pi_2 = 2(q_{s-4}\mathrm{P}^s\mathrm{P}^{s-4} + q_{s+4}\mathrm{P}^s\mathrm{P}^{s+4}) + (q_{s-2}\mathrm{P}^{s-2})^2 + (q_{s+2}\mathrm{P}^{s+2})^2$$
$$+ 2q_{s-2}q_{s+2}\mathrm{P}^{s-2}\mathrm{P}^{s+2}.$$

Then since $\cos\theta d\theta = d\mu$,

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} F_1 d\theta = \int_{-1}^{+1}(\mathrm{P}^s)^2 d\mu + \beta\int_{-1}^{+1}\frac{(\gamma-1)(\mathrm{P}^s)^2}{1-\mu^2}d\mu - 2\beta\int_{-1}^{+1}(q_{s-2}\mathrm{P}^s\mathrm{P}^{s-2} + q_{s+2}\mathrm{P}^s\mathrm{P}^{s+2})d\mu$$
$$+ \beta^2\int_{-1}^{+1}\frac{(\gamma-1)(\mathrm{P}^s)^2}{1-\mu^2}d\mu + \beta^2\int_{-1}^{+1}\frac{(\frac{3}{2}-\gamma)(\mathrm{P}^s)^2}{(1-\mu^2)^2}d\mu$$
$$- 2\beta^2\int_{-1}^{+1}\frac{(\gamma-1)(q_{s-2}\mathrm{P}^s\mathrm{P}^{s-2} + q_{s+2}\mathrm{P}^s\mathrm{P}^{s+2})}{1-\mu^2}d\mu$$
$$+ \beta^2\int_{-1}^{+1}[2q_{s-4}\mathrm{P}^s\mathrm{P}^{s-4}$$
$$+ 2q_{s+4}\mathrm{P}^s\mathrm{P}^{s+4} + (q_{s-2}\mathrm{P}^{s-2})^2 + (q_{s+2}\mathrm{P}^{s+2})^2 + 2q_{s-2}q_{s+2}\mathrm{P}^{s-2}\mathrm{P}^{s+2}]d\mu \quad . \quad . \quad . \quad . \quad (66).$$

And $\int F_2 d\theta$ has the same form, but with accented q's, and with $\gamma + 1$ replacing $\gamma - 1$, and $\gamma - \frac{1}{2}$ replacing $\frac{3}{2} - \gamma$.

It is now necessary to evaluate the several definite integrals involved in this expression.

It is well known that

$$\int_{-1}^{+1} (\mathrm{P}_i^s(\mu))^2 d\mu = \frac{2}{2i+1} \cdot \frac{i+s\,!}{i-s\,!}.$$

It is easy to see that it is possible to express P^{s+2k} in the form

$$\mathrm{P}_i^{s+2k} = \mathrm{AP}_i + \mathrm{BP}_{i-2}^s + \mathrm{CP}_{i-4}^s + \ldots,$$

where A, B, C . . . do not involve μ.

The value of A may be found by considering only the highest power of μ on each side of the identity.

Now $$\mathrm{P}_i^{s+2k} = \frac{(1-\mu^2)^{\frac{1}{2}s+k}}{2^i \,.\, i\,!} \left(\frac{d}{d\mu}\right)^{i+s+2k} (\mu^2-1)^i,$$

$$= (-)^{\frac{1}{2}s+k} \frac{2i\,!}{2^i \,.\, i\,!\; i-s-2k\,!} \mu^i + \ldots ;$$

and $$\mathrm{P}^s = (-)^{\frac{1}{2}s} \frac{2i\,!}{2^i \,.\, i\,!\; i-s\,!} \mu^i + \ldots .$$

Therefore $$\mathrm{A} = (-)^k \frac{i-s\,!}{i-s-2k\,!}.$$

Then, since the integral of the product of two P's of different orders vanishes, we have

$$\int_{-1}^{+1} \mathrm{P}^s \mathrm{P}^{s+2k} d\mu = (-)^k \frac{i-s\,!}{i-s-2k\,!} \int_{-1}^{+1} (\mathrm{P}^s)^2 d\mu = (-)^k \frac{2}{2i+1} \cdot \frac{i+s\,!}{i-s-2k\,!}.$$

We will next consider $\int \frac{\mathrm{P}^s \mathrm{P}^{s+2k}}{1-\mu^2} d\mu$, where k is not zero.

The differential equation gives

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\mathrm{P}^{s+2k}}{d\mu}\right] + i(i+1)\,\mathrm{P}^{s+2k} - \frac{(s+2k)^2}{1-\mu^2}\mathrm{P}^{s+2k} = 0,$$

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\mathrm{P}^s}{d\mu}\right] + i(i+1)\,\mathrm{P}^s - \frac{s^2}{1-\mu^2}\mathrm{P}^s = 0.$$

Multiply the first of these by P^s and the second by P^{s+2k} and subtract, and we have

$$\frac{4k(s+k)\,\mathrm{P}^s\mathrm{P}^{s+2k}}{1-\mu^2} = \mathrm{P}^{s+2k}\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\mathrm{P}^s}{d\mu}\right] - \mathrm{P}^s\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\mathrm{P}^{s+2k}}{d\mu}\right].$$

Therefore*

$$4k(s+k)\int_{-1}^{+1}\frac{P^sP^{s+2k}}{1-\mu^2}d\mu = (1-\mu^2)\left[P^{s+2k}\frac{dP^s}{d\mu} - P^s\frac{dP^{s+2k}}{d\mu}\right], \text{ between limits } \pm 1,$$
$$= 0.$$

Again since by (11)

$$\frac{P^{s+2k}}{1-\mu^2} = AP^{s+2k+2} + BP^{s+2k} + CP^{s+2k-2},$$

it follows that

$$\int_{-1}^{+1}\frac{P^sP^{s+2k}}{(1-\mu^2)^2}d\mu = 0, \text{ unless } k = 0 \text{ or } 1.$$

It remains to find the integrals of $\frac{(P^s)^2}{1-\mu^2}$, $\frac{(P^s)^2}{(1-\mu^2)^2}$, and $\frac{P^sP^{s+2}}{(1-\mu^2)^2}$.

We have seen in (11) (transformed to accord with our present definition of P^s) that

$$\frac{P^s}{1-\mu^2} = \frac{1}{4s(s+1)}P^{s+2} + \tfrac{1}{2}\left[\frac{i(i+1)}{s^2-1}+1\right]P^s + \frac{\{i,s\}\{i,s-1\}}{4s(s-1)}P^{s-2}.$$

Hence

$$\int\frac{(P^s)^2}{1-\mu^2}d\mu = \frac{1}{4s(s+1)}\int P^sP^{s+2}d\mu + \tfrac{1}{2}\left[\frac{i(i+1)}{s^2-1}+1\right]\int(P^s)^2d\mu$$
$$+ \frac{\{i,s\}\{i,s-1\}}{4s(s-1)}\int P^sP^{s-2}d\mu,$$

$$\int\frac{(P^s)^2}{(1-\mu^2)^2}d\mu = \frac{1}{4s(s+1)}\int\frac{P^sP^{s+2}}{1-\mu^2}d\mu + \tfrac{1}{2}\left[\frac{i(i+1)}{s^2-1}+1\right]\int\frac{(P^s)^2}{1-\mu^2}d\mu$$
$$+ \frac{\{i,s\}\{i,s-1\}}{4s(s-1)}\int\frac{P^sP^{s-2}}{1-\mu^2}d\mu,$$

$$\int\frac{P^sP^{s+2}}{(1-\mu^2)^2}d\mu = \frac{1}{4s(s+1)}\int\frac{(P^{s+2})^2}{1-\mu^2}d\mu + \tfrac{1}{2}\left[\frac{i(i+1)}{s^2-1}+1\right]\int\frac{P^sP^{s+2}}{1-\mu^2}d\mu$$
$$+ \frac{\{i,s\}\{i,s-1\}}{4s(s-1)}\int\frac{P^{s-2}P^{s+2}}{1-\mu^2}d\mu.$$

The first of these involves integrals already determined; on introducing them on the right and reducing we find the result to be $\frac{1}{s}\cdot\frac{i+s\,!}{i-s\,!}$.

The first and last terms of the second integral vanish, and the integral is clearly $\tfrac{1}{2}\left[\frac{i(i+1)}{s^2-1}+1\right]\cdot\frac{1}{s}\cdot\frac{i+s\,!}{i-s\,!}$.

The second and third terms of the third integral vanish, and the whole is clearly $\frac{1}{4s(s+1)(s+2)}\,\frac{i+s+2\,!}{i-s-2\,!}$.

* I owe this method of finding these last two integrals to Mr. HOBSON.

Collecting results we have

$$\left.\begin{aligned}
\int_{-1}^{+1} (\mathrm{P}^s)^2\, d\mu &= \frac{2}{2i+1} \cdot \frac{i+s\,!}{i-s\,!} \\
\int_{-1}^{+1} \mathrm{P}^s \mathrm{P}^{s+2k}\, d\mu &= (-)^k \frac{2}{2i+1} \cdot \frac{i+s\,!}{i-s-2k\,!} \\
\int_{-1}^{+1} \frac{\mathrm{P}^s \mathrm{P}^{s+2k}}{1-\mu^2}\, d\mu &= 0. \\
\int_{-1}^{+1} \frac{(\mathrm{P}^s)^2}{1-\mu^2}\, d\mu &= \frac{1}{s} \cdot \frac{i+s\,!}{i-s\,!} \\
\int_{-1}^{+1} \frac{(\mathrm{P}^s)^2}{(1-\mu^2)^2}\, d\mu &= \tfrac{1}{2}\left[\frac{i(i+1)}{s^2-1} + 1\right] \cdot \frac{1}{s} \cdot \frac{i+s\,!}{i-s\,!} \\
\int_{-1}^{+1} \frac{\mathrm{P}^s \mathrm{P}^{s+2}}{(1-\mu^2)^2}\, d\mu &= \frac{1}{4s(s+1)(s+2)} \cdot \frac{i+s+2\,!}{i-s-2\,!} \\
\int_{-1}^{+1} \frac{\mathrm{P}^s \mathrm{P}^{s-2}}{(1-\mu^2)^2}\, d\mu &= \frac{1}{4s(s-1)(s-2)} \cdot \frac{i+s\,!}{i-s\,!}
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (67).$$

Then by means of (66) and (67)

$$\begin{aligned}
\int F_1 d\theta = {} & \frac{2}{2i+1}\, \frac{i+s\,!}{i-s\,!} + \beta(\gamma - 1) \cdot \frac{1}{s} \cdot \frac{i+s\,!}{i-s\,!} \\
& + 2\beta\left(q_{s-2} \frac{2}{2i+1} \cdot \frac{i+s-2\,!}{i-s\,!} + q_{s+2} \cdot \frac{2}{2i+1} \cdot \frac{i+s\,!}{i-s-2\,!}\right) \\
& + \beta^2(\gamma - 1) \cdot \frac{1}{s} \cdot \frac{i+s\,!}{i-s\,!} + \beta^2(\tfrac{3}{2} - \gamma)\tfrac{1}{2}\left[\frac{i(i+1)}{s^2-1} + 1\right] \cdot \frac{1}{s} \cdot \frac{i+s\,!}{i-s\,!} \\
& + \frac{2\beta^2}{2i+1}\left[2q_{s-4}\frac{i+s-4\,!}{i-s\,!} + 2q_{s+4}\frac{i+s\,!}{i-s-4\,!} + (q_{s-2})^2 \frac{i+s-2\,!}{i-s+2\,!}\right. \\
& \qquad\qquad \left. + (q_{s+2})^2 \frac{i+s+2\,!}{i-s-2\,!} + q_{s-2}q_{s+2}\frac{i+s-2\,!}{i-s-2\,!}\right].
\end{aligned}$$

Therefore

$$\begin{aligned}
\int F_1 d\theta = {} & \frac{2}{2i+1} \cdot \frac{i+s\,!}{i-s\,!}\left[1 + \frac{2\beta q_{s-2}}{(i+s)(i+s-1)} + 2\beta q_{s+2}(i-s)(i-s-1)\right. \\
& \qquad + \frac{2\beta^2 q_{s-4}}{(i+s)(i+s-1)(i+s-2)(i+s-3)} \\
& \qquad + 2\beta^2 q^{s+4}(i-s)(i-s-1)(i-s-2)(i-s-3) \\
& \qquad + \frac{\beta^2 (q_{s-2})^2}{(i-s+1)(i-s+2)(i+s)(i+s-1)} \\
& \qquad + \beta^2(q_{s+2})^2(i+s+1)(i+s+2)(i-s)(i-s-1) \\
& \qquad \left. + 2\beta^2 q_{s-2}q_{s+2}\frac{(i-s)(i-s-1)}{(i+s)(i+s-1)}\right] \\
& + \beta \cdot \frac{1}{s}\, \frac{i+s\,!}{i-s\,!}\left\{(1+\beta)(\gamma-1) + \beta(\tfrac{3}{2} - \gamma) \cdot \tfrac{1}{2}\left[\frac{i(i+1)}{s^2-1} + 1\right]\right\}. \quad (68).
\end{aligned}$$

Also $\int F_2 d\theta$ has a similar form with accented q's, $\gamma + 1$ for $\gamma - 1$, and $\gamma - \frac{1}{2}$ for $\frac{3}{2} - \gamma$.

When we substitute for γ its value $1 - \cos 2\phi$, and write as before

$$\Sigma = \frac{i(i+1)}{s^2 - 1},$$

the last term in $\int F_1 d\theta$ becomes

$$+ \beta \cdot \frac{1}{s} \cdot \frac{i+s!}{i-s!} \{\tfrac{1}{4}\beta(\Sigma + 1) - \cos 2\phi [1 - \tfrac{1}{2}\beta(\Sigma - 1)]\} \quad . \quad . \quad . (68).$$

Also the last term in $\int F_2 d\theta$ becomes

$$+ \beta \cdot \frac{1}{s} \cdot \frac{i+s!}{i-s!} \{2[1 + \tfrac{1}{8}\beta(\Sigma + 9)] - \cos 2\phi [1 + \tfrac{1}{2}\beta(\Sigma + 3)]\}. \quad . (68).$$

But it will appear later that we only need the parts of these terms which involve $\cos 2\phi$ developed as far as the first power of β; hence in both cases we may write the latter term inside { } simply as $-\cos 2\phi$.

Our general formulæ for the q coefficients apply for all values of s down to $s = 3$, inclusive, although the result for $s = 3$ needs proper interpretation. Hence the present result applies down to $s = 3$, inclusive.

I have just re-defined Σ, and I remind the reader that

$$\Upsilon = \frac{(i-1)i(i+1)(i+2)}{s^2 - 4}.$$

Then if in (68) we introduce for the q's their values, we find that the coefficient of the term in β is

$$\frac{2q_{s-2}}{(i+s)(i+s-1)} + q_{s+2}(i-s)(i-s-1) = -\tfrac{1}{2}(\Sigma - 1).$$

The coefficient of the term in β^2 is

$$\frac{1}{64}\left\{\frac{(i-s+1)(i-s+2)(i-s+3)(i-s+4)}{(s-1)(s-2)} + \frac{(i-s)(i-s-1)(i-s-2)(i-s-3)}{(s+1)(s+2)}\right.$$
$$+ \frac{(i+s)(i+s-1)(i-s+1)(i-s+2)}{(s-1)^2} + \frac{(i+s+1)(i+s+2)(i-s)(i-s-1)}{(s+1)^2}$$
$$\left. - 2\frac{(i-s+1)(i-s+2)(i-s)(i-s-1)}{(s^2-1)}\right\}.$$

If this be reduced by a process similar to that employed in § 10, we find

$$\int F_1 d\theta = \frac{2}{2i+1}\frac{i+s!}{i-s!}\{1-\tfrac{1}{2}\beta(\Sigma-1)+\tfrac{1}{32}\beta^2[3\Sigma^2-6\Sigma+6-s^2(\Sigma^2+2\Sigma-1)+2\Upsilon]\}$$
$$+\beta\cdot\frac{1}{s}\cdot\frac{i+s!}{i-s!}\{\tfrac{1}{4}\beta(\Sigma+1)-\cos 2\phi\} \quad . (69).$$

We know that P^s is derivable from $\mathfrak{P}^s$ by multiplication by $1/C_i^s$, and we have found in (33), § 9,

$$\frac{1}{(C_i^s)^2} = 1+\beta(\Sigma+1)+\tfrac{1}{8}\beta^2[3\Sigma^2+8\Sigma+5+s^2\Sigma^2-\Upsilon].$$

Hence multiplying (69) by $\frac{1}{(C_i^s)^2}$ we have

$$\int F_2 d\theta = \frac{2}{2i+1}\frac{i+s!}{i-s!}\{1+\tfrac{1}{2}\beta(\Sigma+3)+\tfrac{1}{32}\beta^2[-\Sigma^2+26\Sigma+42$$
$$+s^2(3\Sigma^2-2\Sigma+1)-2\Upsilon]\}+\beta\cdot\frac{1}{s}\cdot\frac{i+s!}{i-s!}\{\tfrac{1}{4}\beta(\Sigma+1)-\cos 2\phi\} \quad . (70).$$

I have also obtained this result by direct development. It may be thought surprising that the last term is now the same in both formulæ, notwithstanding the difference in the earlier stages, but if the reader will go through the analysis he will see how this has been brought about. The formulæ (69) and (70) also hold true when $s=3$ (as I have verified), notwithstanding the fact that P^3 is not to be derived from $\mathfrak{P}^3$ by a factor.

The next step is the integration with respect to ϕ.

We have

$$\begin{Bmatrix}\mathfrak{C}^s\\ \mathfrak{S}^s\end{Bmatrix} = \begin{Bmatrix}\cos\\ \sin\end{Bmatrix} s\phi + \beta\left[p_{s-2}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(s-2)\phi + p_{s+2}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(s+2)\phi\right]$$
$$+\beta^2\left[p_{s-4}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(s-4)\phi + p_{s+4}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(s+4)\phi\right].$$

Therefore

$$\begin{Bmatrix}(\mathfrak{C}_i^s)^2\\ (\mathfrak{S}_i^s)^2\end{Bmatrix} = \tfrac{1}{2}\pm\tfrac{1}{2}\cos 2s\phi+\beta[(p_{s-2}+p_{s+2})\cos 2\phi \pm p_{s-2}\cos 2(s-1)\phi$$
$$\pm p_{s+2}\cos 2(s+1)\phi]+\beta^2[\tfrac{1}{2}(p_{s-2})^2+\tfrac{1}{2}(p_{s+2})^2+(p_{s-4}+p_{s+4}$$
$$+p_{s-2}p_{s+2})\cos 4\phi \pm p_{s-2}p_{s+2}\cos 2s\phi \pm (p_{s-4}+\tfrac{1}{2}(p_{s-2})^2)\cos 2(s-2)\phi$$
$$\pm(p_{s+4}+\tfrac{1}{2}(p_{s+2})^2)\cos 2(s+2)\phi].$$

Also $\dfrac{(\mathsf{C}_i^s)^2 \text{ or } (\mathsf{S}_i^s)^2}{1-\beta\cos 2\phi}$ have the same forms with accented p's.

Accordingly, with unaccented p's, we have to multiply this expression by

$(1 - \beta \cos 2\phi)^{-\frac{1}{2}}$, and with accented p's we multiply by $(1 - \beta \cos 2\phi)^{\frac{1}{2}}$, and we shall then have the functions denoted above by K^2.

The function K^2 has to be multiplied by a function of the form $A + B\beta \cos 2\phi$, and integrated from $\phi = 2\pi$ to 0. It follows that the only terms in K^2 which will not vanish are those independent of ϕ and those in $\cos 2\phi$; moreover, the latter terms are only required as far as the first power of β.

Now
$$(1 - \beta \cos 2\phi)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}\beta \cos 2\phi + \tfrac{3}{16}\beta^2 (1 + \cos 4\phi),$$
$$(1 - \beta \cos 2\phi)^{\frac{1}{2}} = 1 - \tfrac{1}{2}\beta \cos 2\phi - \tfrac{1}{16}\beta^2 (1 + \cos 4\phi).$$

Then omitting terms which will vanish on integration

$$\frac{(\mathfrak{C}_i^s \text{ or } \mathfrak{S}_i^s)^2}{(1 - \beta \cos 2\phi)^{\frac{1}{2}}} = \tfrac{1}{2}\{1 + \beta^2 [(p_{s-2})^2 + (p_{s+2})^2 + \tfrac{1}{2}p_{s-2} + \tfrac{1}{2}p_{s+2} + \tfrac{3}{16}]\}$$
$$+ \beta(p_{s-2} + p_{s+2} + \tfrac{1}{4}) \cos 2\phi,$$
$$\frac{(\mathsf{C}_i^s \text{ or } \mathsf{S}_i^s)^2}{(1 - \beta \cos 2\phi)^{\frac{1}{2}}} = \tfrac{1}{2}\{1 + \beta^2 [(p'_{s-2})^2 + (p'_{s+2})^2 - \tfrac{1}{2}p'_{s-2} - \tfrac{1}{2}p'_{s+2} - \tfrac{1}{16}]\}$$
$$+ \beta(p'_{s-2} + p'_{s+2} - \tfrac{1}{4}) \cos 2\phi.$$

However, the latter formula is not needed except for verification, because it will be derivable from the former by multiplication by $\frac{1}{(D_i^s)^2}$.

Now if we substitute for the p's their values as given in (27), § 8, we find

$$\frac{(\mathfrak{C}_i^s \text{ or } \mathfrak{S}_i^s)^2}{(1 - \beta \cos 2\phi)^{\frac{1}{2}}} = \tfrac{1}{2}\{1 + \tfrac{1}{32}\beta^2 [\Sigma^2 + 4\Sigma + 6 + s^2(\Sigma^2 - 2\Sigma + 1)]\} + \tfrac{1}{4}\beta(\Sigma + 1) \cos 2\phi.$$

And multiplying by $\frac{1}{(D_i^s)^2}$ or $1 - \frac{1}{8}\beta^2(\Sigma + 3)$, or developing directly

$$\frac{(\mathsf{C}_i^s \text{ or } \mathsf{S}_i^s)^2}{(1 - \beta \cos 2\phi)^{\frac{1}{2}}} = \tfrac{1}{2}\{1 + \tfrac{1}{32}\beta^2 [\Sigma^2 - 6 + s^2(\Sigma^2 - 2\Sigma + 1)]\} + \tfrac{1}{4}\beta(\Sigma + 1) \cos 2\phi.$$

These represent the K^2 of our integrals.

Then

$$\int p \left(\mathfrak{P}_i^s \begin{Bmatrix} \mathfrak{C}_i^s \\ \mathfrak{S}_i^s \end{Bmatrix}\right)^2 d\sigma = M(1 - \beta)^{\frac{1}{2}} \iint F_1 \frac{(\mathfrak{C}^s \text{ or } \mathfrak{S}^s)^2}{(1 - \beta \cos 2\phi)^{\frac{1}{2}}} \, d\theta d\phi,$$
$$= \frac{2\pi M (1 - \beta)^{\frac{1}{2}}}{2i + 1} \cdot \frac{i + s\,!}{i - s\,!} \{1 - \tfrac{1}{2}\beta(\Sigma - 1)$$
$$+ \tfrac{1}{32}\beta^2 [3\Sigma^2 - 6\Sigma + 6 - s^2(\Sigma^2 + 2\Sigma - 1) + 2\Upsilon]\}$$
$$\times \{1 + \tfrac{1}{32}\beta^2 [\Sigma^2 + 4\Sigma + 6 + s^2(\Sigma^2 - 2\Sigma + 1)]\}$$
$$+ \tfrac{1}{4}\pi M (1 - \beta)^{\frac{1}{2}} \left[\beta^2 \cdot \frac{1}{s} \cdot \frac{i + s\,!}{i - s\,!} (\Sigma + 1) - \beta^2 \cdot \frac{1}{s} \cdot \frac{i + s\,!}{i - s\,!} (\Sigma + 1)\right].$$

$$= \frac{2\pi M}{2i+1}\frac{i+s\,!}{i-s\,!}(1-\beta)^{\frac{1}{2}}\{1-\tfrac{1}{2}\beta(\Sigma-1)$$
$$+\tfrac{1}{16}\beta^2[2\Sigma^2-\Sigma+6-s^2(2\Sigma-1)+\Upsilon]\}$$
$$= \frac{2\pi M}{2i+1}\frac{i+s\,!}{i-s\,!}\{1-\tfrac{1}{2}\beta\Sigma+\tfrac{1}{16}\beta^2[2\Sigma^2+3\Sigma-s^2(2\Sigma-1)+\Upsilon]\} \quad . \quad . \quad . \quad . \quad . \quad (71).$$

The following results may be obtained either by direct development, or by multiplication by either or both the factors $\frac{1}{(C_i^s)^2}$ and $\frac{1}{(D_i^s)^2}$. The former converts $\mathfrak{P}$ into P, the latter $\mathfrak{C}$ or $\mathfrak{S}$ into C or S.

$$\left.\begin{aligned}
&\int p\left(\mathfrak{P}_i^s\left\{\begin{matrix}\mathsf{C}_i^s\\ \mathsf{S}_i^s\end{matrix}\right)^2 d\sigma = \frac{2\pi M}{2i+1}\frac{i+s\,!}{i-s\,!}\{1-\tfrac{1}{2}\beta\Sigma \\
&\qquad\qquad +\tfrac{1}{16}\beta^2[2\Sigma^2+\Sigma-6-s^2(2\Sigma-1)+\Upsilon]\}, \\
&\int p\left(\mathsf{P}_i^s\left\{\begin{matrix}\mathfrak{C}_i^s\\ \mathfrak{S}_i^s\end{matrix}\right)^2 d\sigma = \quad . \quad . \quad . \quad . \quad \{1+\tfrac{1}{2}\beta(\Sigma+2) \\
&\qquad\qquad +\tfrac{1}{16}\beta^2[11\Sigma+10+s^2(2\Sigma^2-2\Sigma+1)-\Upsilon]\}, \\
&\int p\left(\mathsf{P}_i^s\left\{\begin{matrix}\mathsf{C}_i^s\\ \mathsf{S}_i^s\end{matrix}\right)^2 d\sigma = \quad . \quad . \quad . \quad . \quad \{1+\tfrac{1}{2}\beta(\Sigma+2) \\
&\qquad\qquad +\tfrac{1}{16}\beta^2[9\Sigma+4+s^2(2\Sigma^2-2\Sigma+1)-\Upsilon]\}
\end{aligned}\right\} \quad (71).$$

§ 15. *Integration in the case of* $s = 2$.

Although the development in powers of $\sec^2\theta$ is still legitimate in this case, yet the formulæ found in the last section fail because Υ contains s^2-4 in the denominator. Moreover since $\mathfrak{P}^2$ is not convertible into P^2 by a factor each case must be considered separately.

We now have $q_{s-4}=0$, $q'_{s-4}=0$, and therefore from (68)

$$\int F_1 d\theta = \frac{2}{2i+1}\frac{i+2\,!}{i-2\,!}\Big\{1+2\beta\Big[\frac{q_0}{(i+1)(i+2)}+q_4(i-2)(i-3)\Big]$$
$$+\beta^2\Big[2q_6(i-2)(i-3)(i-4)(i-5)+\frac{(q_0)^2}{(i-1)\,i\,(i+1)(i+2)}$$
$$+(q_4)^2(i+3)(i+4)(i-2)(i-3)+2q_0q_4\frac{(i-2)(i-3)}{(i+1)(i+2)}\Big]\Big\}$$
$$+\tfrac{1}{2}\beta\frac{i+2\,!}{i-2\,!}\{\tfrac{1}{4}\beta(\Sigma+1)-\cos 2\phi\}.$$

$\int F_2 d\theta$ is equal to the same with accented q's, and the last term equal to

$$\tfrac{1}{2}\beta\frac{i+2\,!}{i-2\,!}\{2+\tfrac{1}{4}\beta(\Sigma+9)-\cos 2\phi\}.$$

We now have

$q_0 = -\frac{1}{8}\{i, 1\}\{i, 2\}$, $q_4 = \frac{1}{24}$, $q_6 = \frac{1}{1536}$, associated with a cosine function.
$q'_0 = \quad 0 \quad$, $q'_4 = \frac{1}{12}$, $q'_6 = \frac{1}{512}$, associated with a sine function.

It is well to note that these values are given by the general formula, because this consideration shows that much of the previous reductions is still applicable.

Effecting the reductions I find

$$\int F_1 d\theta = \frac{2}{2i+1} \cdot \frac{i+2\,!}{i-2\,!}\{1 - \tfrac{1}{2}\beta(\Sigma - 1) + \tfrac{1}{64}\beta^2[\tfrac{19}{4}\Sigma^2 - \tfrac{65}{2}\Sigma + 20 + 2(2i+1)(3\Sigma - 1)]\} + \tfrac{1}{2}\beta\frac{i+2\,!}{i-2\,!}\{\tfrac{1}{4}\beta(\Sigma+1) - \cos 2\phi\},$$

$$= \frac{2}{2i+1}\frac{i+2\,!}{i-2\,!}\{1 - \tfrac{1}{2}\beta(\Sigma - 1) + \tfrac{1}{256}\beta^2(19\Sigma^2 - 130\Sigma + 80)\} + \tfrac{1}{2}\beta\frac{i+2\,!}{i-2\,!}\{\tfrac{1}{8}\beta(5\Sigma + 1) - \cos 2\phi\}.$$

This integral will be associated with $\mathfrak{C}_i^2$ and C_i^2, and in the present case $\Sigma = \frac{1}{3}i(i+1)$.

In the same way

$$\int F_2 d\theta = \frac{2}{2i+1}\frac{i+2\,!}{i-2\,!}\{1 + \tfrac{1}{2}\beta[\Sigma + 3 - (2i+1)] + \tfrac{1}{64}\beta^2[\tfrac{25}{4}\Sigma^2 + \tfrac{93}{2}\Sigma + 92 - 6(2i+1)(\Sigma+5)]\} + \tfrac{1}{2}\beta\frac{i+2\,!}{i-2\,!}\{2 + \tfrac{1}{4}\beta(\Sigma+9) - \cos 2\phi\},$$

$$= \frac{2}{2i+1}\frac{i+2\,!}{i-2\,!}\{1 + \tfrac{1}{2}\beta(\Sigma+3) + \tfrac{1}{256}\beta^2(25\Sigma^2 + 186\Sigma + 368)\} + \tfrac{1}{2}\beta\frac{i+2\,!}{i-2\,!}\{-\tfrac{1}{8}\beta(\Sigma - 3) - \cos 2\phi\}.$$

This will be associated with $\mathfrak{S}_i^2$ and S_i^2.

Now turning to the cosine and sine functions, we find that they must be treated apart, but the integral involving C_i^2 may be derived from that in $\mathfrak{C}_i^2$ by the factor $\left[\frac{1}{D_i^2}(\cos)\right]^2$; and similarly S_i^2 from $\mathfrak{S}_i^2$ by the factor $\left[\frac{1}{D_i^2}(\sin)\right]^2$. These factors were evaluated in (36), § 9.

We now have $p_{s-4} = 0$, $p'_{s-4} = 0$; also for the sine function $p_{s-2} = p_0 = 0$. Then

$$(\mathfrak{C}_i^2)^2 = \tfrac{1}{2} + \tfrac{1}{2}\cos 4\phi + \beta[(2p_0 + p_4)\cos 2\phi + p_4\cos 6\phi] + \beta^2[(p_0)^2 + \tfrac{1}{2}(p_4)^2 + (p_6 + p_0 p_4)\cos 4\phi + p_0 p_4 \cos 4\phi + (p_6 + \tfrac{1}{2}(p_4)^2)\cos 8\phi],$$

$$(\mathfrak{S}_i^2)^2 = \tfrac{1}{2} - \tfrac{1}{2}\cos 4\phi + \beta[p_4\cos 2\phi - p_4\cos 6\phi] + \beta^2[\tfrac{1}{2}(p_4)^2 + p_6\cos 4\phi - (p_6 + \tfrac{1}{2}(p_4)^2)\cos 8\phi].$$

Then as far as material

$$\frac{(\mathfrak{C}_i^2)^2}{(1-\beta\cos 2\phi)^{\frac12}} = \tfrac12\{1+\beta^2[(p_4)^2+2(p_0)^2+p_0+\tfrac12 p_4+\tfrac{9}{32}]\} + \beta(p_4+2p_0+\tfrac38)\cos 2\phi\,,$$

$$\frac{(\mathfrak{S}_i^2)^2}{(1-\beta\cos 2\phi)^{\frac12}} = \tfrac12\{1+\beta^2[(p_4)^2+\tfrac12 p_4+\tfrac{3}{32}]\} + \beta(p_4+\tfrac18)\cos 2\phi\,.$$

Now $\qquad p_0 = \tfrac18\{i, 2\}, \quad p_4 = -\tfrac{1}{24}\{i, 3\} \quad$ and

$$\frac{(\mathfrak{C}_i^2)^2}{(1-\beta\cos 2\phi)^{\frac12}} = \tfrac12\{1+\tfrac{1}{64}\beta^2(19\Sigma^2-8\Sigma+22)\} + \tfrac18\beta(5\Sigma+1)\cos 2\phi\,,$$

$$\frac{(\mathfrak{S}_i^2)^2}{(1-\beta\cos 2\phi)^{\frac12}} = \tfrac12\{1+\tfrac{1}{64}\beta^2(\Sigma^2-8\Sigma+18)\} - \tfrac18\beta(\Sigma-3)\cos 2\phi\,.$$

We now multiply these by $\int F_1 d\theta$ and $\int F_2 d\theta$ respectively, and the last terms disappear as before. I remark that the disappearance of the terms which do not involve the factor $1/(2i+1)$ affords an excellent test of the correctness of the laborious reductions throughout all this part of the work.

Then we have

$$\int p(\mathfrak{P}_i^2\mathfrak{C}_i^2)^2 d\sigma = \frac{2\pi M}{2i+1}\frac{i+2!}{i-2!}(1-\beta)^{\frac12}[1-\tfrac12\beta(\Sigma-1)+\tfrac{1}{256}\beta^2(19\Sigma^2-130\Sigma+80)] \times[1+\tfrac{1}{64}\beta^2(19\Sigma^2-8\Sigma+22)]$$

$$= \frac{2\pi M}{2i+1}\frac{i+2!}{i-2!}\{1-\tfrac12\beta\Sigma+\tfrac{1}{256}\beta^2(95\Sigma^2-98\Sigma+72)\} \quad . \quad . \quad . \quad . \quad . \quad (72).$$

If we multiply this by $\left[\frac{1}{D_i^2}(\cos)\right]^2$ or $1-\frac{1}{16}\beta^2(5\Sigma+7)$, we obtain the result when $\mathbf{C}_i^2$ replaces $\mathfrak{C}_i^2$; the only change is that the last term inside { } now becomes $+\frac{1}{256}\beta^2(95\Sigma^2-178\Sigma-40)$.

Again

$$\int p(\mathsf{P}_i^2\mathfrak{S}_i^2)^2 d\sigma = \frac{2\pi M}{2i+1}\frac{i+2!}{i-2!}(1-\beta)^{\frac12}[1+\tfrac12\beta(\Sigma+3)+\tfrac{1}{256}\beta^2(25\Sigma^2+186\Sigma+368)] \times[1+\tfrac{1}{64}\beta^2(\Sigma^2-8\Sigma+18)]$$

$$= \frac{2\pi M}{2i+1}\frac{i+2!}{i-2!}[1+\tfrac12\beta(\Sigma+2)+\tfrac{1}{256}\beta^2(29\Sigma^2+90\Sigma+216)] \quad . \quad . \quad . \quad . \quad (72).$$

If we multiply this by $\left[\frac{1}{D_i^2}(\sin)\right]^2$ or $1+\frac{1}{16}\beta^2(\Sigma+5)$, we obtain the result when $\mathbf{S}_i^2$ replaces $\mathfrak{S}_i^2$; the only change is that the last term inside { } now becomes $+\frac{1}{256}\beta^2(29\Sigma^2+106\Sigma+136)$.

This terminates the integrals, which can be completely determined by this method of developing in powers of $\sec^2\theta$.

§ 16. *Portion of the Integration in the case of s = 1.*

The preceding method may be used for finding the four integrals

$$\int p \begin{Bmatrix} (\mathfrak{S}_i^1)^2 \\ (\mathsf{S}_i^1)^2 \end{Bmatrix} [(\mathfrak{P}_i^1)^2 - (\mathrm{P}_i^1)^2] d\sigma$$

and

$$\int p \begin{Bmatrix} (\mathfrak{C}_i^1)^2 \\ (\mathsf{C}_i^1)^2 \end{Bmatrix} \left[(\mathsf{P}_i^1)^2 - \left(\mathrm{P}_i^1 \sqrt{\frac{\frac{1+\beta}{1-\beta} - \sin^2\theta}{\cos^2\theta}} \right)^2 \right] d\theta.$$

There will then remain four integrals of the type $\int p\,(\mathfrak{S}_i^1)^2\,(\mathrm{P}_i^1)^2\,d\sigma$ to evaluate.

The first pair of our integrals are clearly to be treated by putting $\Pi_0 = 0$, $q_{s-2} = q_{-1} = 0$, $q_{s-4} = q_{-3} = 0$, and then determining $\int \mathrm{F}_1 d\theta$. The condition for the second pair only differs in the accentuation of the q's which vanish, and in the use of $\int \mathrm{F}_2 d\theta$.

The vanishing of Π_0 makes

$$\mathrm{F}_1 = \beta\Pi_1 \cos\theta + \beta^2 \left(\frac{\Pi_1(\gamma - 1)}{\cos\theta} + \Pi_2 \cos\theta \right),$$

$$\mathrm{F}_2 = \beta\Pi_1 \cos\theta + \beta^2 \left(\frac{\Pi_1(\gamma + 1)}{\cos\theta} + \Pi_2 \cos\theta \right).$$

In the first of these

$$\Pi_1 = -2q_3 \mathrm{P}^1\mathrm{P}^3, \quad \Pi_2 = 2q_5\mathrm{P}^1\mathrm{P}^5 + (q_3)^2(\mathrm{P}^3)^2;$$

and in the second the form is the same with accented q's.

Also since $\int \frac{\mathrm{P}^1\mathrm{P}^3}{1-\mu^2} d\mu = 0$, we have $\int \frac{\Pi_1 d\theta}{\cos\theta} = 0$.

Hence

$$\int \mathrm{F}_1 d\theta = -2\beta q_3 \int \mathrm{P}^1\mathrm{P}^3 d\mu + 2\beta^2 \int [(q_5\mathrm{P}^1\mathrm{P}^5 + \tfrac{1}{2}(q_3)^2(\mathrm{P}^3)^2] d\mu,$$

$$= \frac{2}{2i+1} \frac{i+1\,!}{i-1\,!} [2\beta q_3 (i-1)(i-2) + 2\beta^2 q_5 (i-1)(i-2)(i-3)(i-4)$$
$$+ \beta^2 (q_3)^2 (i+2)(i+3)(i-1)(i-2)];$$

and $\int \mathrm{F}_2 d\theta$ is the same with accented q's.

It is only necessary to pursue the cases $\int p\,(\mathfrak{S}_i^1)^2 \mathrm{F}_1 d\sigma$ and $\int p\,(\mathfrak{C}_i^1)^2 \mathrm{F}_2 d\sigma$, since the other pair of integrals may be determined by means of multiplication by the appropriate factors, determined in § 9.

Now for $\mathfrak{P}_i^1 \mathfrak{S}_i^1$, associated with F_1,

$$q_3 = \tfrac{1}{16}[1 - \tfrac{1}{16}\beta i(i+1)], \; q_5 = \tfrac{1}{768};$$

and for $\mathsf{P}_i^1 \mathfrak{C}_i^1$, associated with F_2,

$$q_3' = \tfrac{3}{16}[1 + \tfrac{1}{16}\beta i(i+1)], \; q_5' = \tfrac{5}{768}.$$

Therefore

$$\int F_1 d\theta = \frac{2}{2i+1}\frac{i+1\,!}{i-1\,!}\Big\{\tfrac{1}{8}\beta(i-1)(i-2) + \tfrac{1}{128}\beta^2[\tfrac{1}{3}(i-1)(i-2)(i-3)(i-4) + \tfrac{1}{2}(i+2)(i+3)(i-1)(i-2) - (i-1)(i-2)i(i+1)]\Big\},$$

$$\int F_2 d\theta = \frac{2}{2i+1}\frac{i+1\,!}{i-1\,!}\Big\{\tfrac{3}{8}\beta(i-1)(i-2) + \tfrac{1}{128}\beta^2[\tfrac{5}{3}(i-1)(i-2)(i-3)(i-4) + \tfrac{9}{2}(i+2)(i+3)(i-1)(i-2) + 3(i-1)(i-2)i(i+1)]\Big\}.$$

In the present case we cannot use Σ as an abridgement, since it is infinite; I therefore now write

$$j = i(i+1).$$

Effecting the reductions we have

$$\int F_1 d\theta = \frac{2}{2i+1}\frac{i+1\,!}{i-1\,!}\{\tfrac{1}{8}\beta(j - 2(2i+1) + 4) + \tfrac{1}{128}\beta^2[-\tfrac{1}{6}j^2 + \tfrac{28}{3}j + 30 - 16(2i+1)]\},$$

$$\int F_2 d\theta = \frac{2}{2i+1}\frac{i+1\,!}{i-1\,!}\{\tfrac{3}{8}\beta(j - 2(2i+1) + 4) + \tfrac{1}{128}\beta^2[\tfrac{55}{6}j^2 + \tfrac{188}{3}j + 174 - 16(2i+1)(j+5)]\}.$$

The former of these is associated with $\mathfrak{S}$, the latter with $\mathfrak{C}$.

In the cosine and sine functions we have

$$p_{s-2} = p_{-1} = 0, \qquad p_{s-1} = p_{-3} = 0, \qquad \text{and}$$

$$(\mathfrak{S}_i^1)^2 = \tfrac{1}{2} - \tfrac{1}{2}\cos 2\phi + \beta(p_3 \cos 2\phi - p_3 \cos 4\phi) + \beta^2[\tfrac{1}{2}(p_3)^2 + p_5 \cos 4\phi - (p_5 + \tfrac{1}{2}(p_3)^2)\cos 6\phi],$$

$$(\mathfrak{C}_i^1)^2 = \tfrac{1}{2} + \tfrac{1}{2}\cos 2\phi + \beta(p_3 \cos 2\phi + p_3 \cos 4\phi) + \beta^2[\tfrac{1}{2}(p_3)^2 + p_5 \cos 4\phi + (p_5 + \tfrac{1}{2}(p_3)^2)\cos 6\phi].$$

As far as material, we then have

$$\frac{(\mathfrak{S}_i^1)^2}{(1-\beta\cos 2\phi)^{\frac{1}{2}}} = \tfrac{1}{2}\{1 - \tfrac{1}{4}\beta + \beta^2[(p_3)^2 + \tfrac{1}{2}p_3 + \tfrac{3}{16}]\} + (-\tfrac{1}{2} + \tfrac{1}{4}\beta + \beta p_3)\cos 2\phi,$$

$$= \tfrac{1}{2}\{1 - \tfrac{1}{4}\beta + \beta^2[(p_3)^2 + \tfrac{1}{2}p_3 + \tfrac{3}{16}]\}\{1 - (1 - 2\beta p_3 - \tfrac{1}{4}\beta)\cos 2\phi\};$$

$$\frac{(\mathfrak{C}_i^1)^2}{(1-\beta\cos 2\phi)^{\frac{1}{2}}} = \tfrac{1}{2}\{1+\tfrac{1}{4}\beta+\beta^2[(p_3)^2+\tfrac{1}{2}p_3+\tfrac{3}{16}]\} + (\tfrac{1}{2}+\tfrac{1}{4}\beta+\beta p_3)\cos 2\phi,$$

$$= \tfrac{1}{2}\{1+\tfrac{1}{4}\beta+\beta^2[(p_3)^2+\tfrac{1}{2}p_3+\tfrac{3}{16}]\}\{1+(1+2\beta p_3+\tfrac{1}{4}\beta)\cos 2\phi\}.$$

Now both for sines and cosines, to the order necessary for our present purpose, $p_3 = -\frac{1}{16}\{i, 2\}$. Therefore, introducing j for $i(i+1)$,

$$\left.\begin{aligned}\frac{(\mathfrak{S}_i^1)^2}{(1-\beta\cos 2\phi)^{\frac{1}{2}}} &= \tfrac{1}{2}\{1-\tfrac{1}{4}\beta+\tfrac{1}{256}\beta^2(j^2-12j+68)\}\{1-[1+\tfrac{1}{8}\beta j-\tfrac{1}{2}\beta]\cos 2\phi\}\\ \frac{(\mathfrak{C}_i^1)^2}{(1-\beta\cos 2\phi)^{\frac{1}{2}}} &= \tfrac{1}{2}\{1+\tfrac{1}{4}\beta+\tfrac{1}{256}\beta^2(j^2-12j+68)\}\{1+[1-\tfrac{1}{8}\beta j+\tfrac{1}{2}\beta]\cos 2\phi\}\end{aligned}\right\} \quad . \quad (73).$$

Observe that $\int F_1 d\theta$ and $\int F_2 d\theta$ do not involve $\cos 2\phi$, and are of the first order in β. Hence, as far as material for the *present* portion of the work,

$$\frac{(\mathfrak{S}_i^1)^2}{(1-\beta\cos 2\phi)^{\frac{1}{2}}} = \tfrac{1}{2}(1-\tfrac{1}{4}\beta), \qquad \frac{(\mathfrak{C}_i^1)^2}{(1-\beta\cos 2\phi)^{\frac{1}{2}}} = \tfrac{1}{2}(1+\tfrac{1}{4}\beta) \quad . \quad . \quad . \quad . \quad (73).$$

Also, to the first order, from (37),

$$\left[\frac{1}{D_i^1}(\sin)\right]^2 = 1+\tfrac{1}{2}\beta, \qquad \left[\frac{1}{D_i^1}(\cos)\right]^2 = 1-\tfrac{1}{2}\beta.$$

Therefore as far as necessary

$$\frac{(\mathbf{S}_i^1)^2}{(1-\beta\cos 2\phi)^{\frac{1}{2}}} = \tfrac{1}{2}(1+\tfrac{1}{4}\beta) \qquad \frac{(\mathbf{C}_i^1)^2}{(1-\beta\cos 2\phi)^{\frac{1}{2}}} = \tfrac{1}{2}(1-\tfrac{1}{4}\beta).$$

Hence

$$\int p\,(\mathfrak{S}_i^1)^2\,[(\mathfrak{P}_i^1)^2-(P_i^1)^2]\,d\sigma = \frac{2\pi M}{2i+1}\,\frac{i+1\,!}{i-1\,!}(1-\beta)^{\frac{1}{2}}(1-\tfrac{1}{4}\beta)\{\tfrac{1}{8}\beta(j-2(2i+1)+4)$$
$$+\tfrac{1}{128}\beta^2[-\tfrac{1}{6}j^2+\tfrac{28}{3}j+30-16(2i+1)]\},$$
$$= \frac{2\pi M}{2i+1}\,\frac{i+1\,!}{i-1\,!}\{\tfrac{1}{8}\beta(j+4)-\tfrac{1}{768}\beta^2(j^2+16j+108)\}$$
$$+\pi M\frac{i+1\,!}{i-1\,!}(-\tfrac{1}{2}\beta+\tfrac{1}{8}\beta^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74).$$

For $\mathbf{S}_i^1$ we have only to replace the factor $1-\frac{1}{4}\beta$ by $1+\frac{1}{4}\beta$, and find

$$\int p(\mathbf{S}_i^1)^2\,[(\mathfrak{P}_i^1)^2-(P_i^1)^2]\,d\sigma = \frac{2\pi M}{2i+1}\,\frac{i+1\,!}{i-1\,!}\{\tfrac{1}{8}\beta(j+4)-\tfrac{1}{768}\beta^2(j^2-32j-84)\}$$
$$+\pi M\frac{i+1\,!}{i-1\,!}(-\tfrac{1}{2}\beta-\tfrac{1}{8}\beta^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74).$$

Again, omitting intermediate steps,

$$\int p\,(\mathfrak{C}_i^1)^2\left[(\mathsf{P}_i^1)^2-\left(\mathrm{P}_i^1\sqrt{\frac{\frac{1+\beta}{1-\beta}-\sin^2\theta}{\cos^2\theta}}\right)^2\right]d\sigma$$

$$=\frac{2\pi\mathrm{M}}{2i+1}\,\frac{i+1\,!}{i-1\,!}\left\{\tfrac{3}{8}\beta\,(j+4)+\tfrac{1}{768}\beta^2\,(55j^2+304j+756)\right\}$$

$$-\pi\mathrm{M}\frac{i+1\,!}{i-1\,!}\left[\tfrac{3}{2}\beta+\tfrac{1}{8}\beta^2(2j+7)\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74).$$

$$\int p(\mathsf{C}_i^1)^2\left[(\mathsf{P}_i^1)^2-\left(\mathrm{P}_i^1\sqrt{\frac{\frac{1+\beta}{1-\beta}-\sin^2\theta}{\cos^2\theta}}\right)^2\right]d\sigma$$

$$=\frac{2\pi\mathrm{M}}{2i+1}\,\frac{i+1\,!}{i-1\,!}\left\{\tfrac{3}{8}\beta\,(j+4)+\tfrac{1}{768}\beta^2(55j^2+160j+180)\right\}$$

$$-\pi\mathrm{M}\frac{i+1\,!}{i-1\,!}\left[\tfrac{3}{2}\beta+\tfrac{1}{8}\beta^2\,(2j+1)\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74).$$

§ 17. *Portion of the Integration in the case of* $s=0$.

We are to find $\int p\left\{\begin{matrix}(\mathfrak{C}_i)^2\\(\mathsf{C}_i)^2\end{matrix}\right.[(\mathfrak{P}_i)^2-(\mathrm{P}_i)^2]\,d\sigma$, leaving two integrals of the type $\int p\,(\mathfrak{C}_i)^2\,(\mathrm{P}_i)^2 d\sigma$ to be determined subsequently.

It is only necessary to consider $\mathfrak{C}_i$, since the other case is determinable from it by multiplication to $\frac{1}{(\mathrm{D}_i)^2}$, as found in (38) of § 9.

Following the procedure of the case where $s=1$, we have

$$\int \mathrm{F}_1 d\theta=-2\beta q_2\int \mathrm{PP}^2 d\mu+2\beta^2\int[q_4\mathrm{PP}^4+\tfrac{1}{2}\,(q_2)^2(\mathrm{P}^2)^2]\,d\mu,$$

$$=\frac{2}{2i+1}[2\beta q_2 i\,(i-1)+2\beta^2 q_4 i\,(i-1)\,(i-2)\,(i-3)+\beta^2\,(q_2)^2(i+1)\,(i+2)\,i(i-1)].$$

Then since $q_2=\frac{1}{4}$, $q_4=\frac{1}{128}$,

$$\int \mathrm{F}_1 d\theta=\frac{2}{2i+1}\left\{\tfrac{1}{2}\beta[j+1-(2i+1)]+\tfrac{1}{64}\beta^2\,[5j^2+14j+12-4\,(2i+1)\,(j+3)]\right\}.$$

Now $\qquad (\mathfrak{C}_i)^2=1+2\beta p_2\cos 2\phi+\beta^2\,[(2p_4+\tfrac{1}{2}(p_2)^2)\cos 4\phi+\tfrac{1}{2}\,(p_2)^2],$

and as far as material

$$\frac{(\mathfrak{C}_i)^2}{(1-\beta\cos 2\phi)^{\frac{1}{2}}}=1+\beta^2\,[\tfrac{1}{2}p_2+\tfrac{1}{2}(p_2)^2+\tfrac{3}{16}]+\beta\,(\tfrac{1}{2}+2p_2)\cos 2\phi,$$

$$=\{1+\tfrac{1}{32}\beta^2(j^2-4j+6)\}\{1-\tfrac{1}{2}\beta\,(j-1)\cos 2\phi\}, \quad . \quad . \quad (75),$$

since $\qquad p_2=-\tfrac{1}{4}i\,(i+1)=-\tfrac{1}{4}j.$

At present we only require this to the first power of β, and since $\int F_1 d\theta$ does not contain $\cos 2\phi$, the expression (75) as far as at present needed is simply unity.

Again, by (38) of § 9,

$$\frac{1}{(\text{D}_i)^2}=1+\tfrac{1}{8}\beta^2(j-3),$$

therefore by multiplication

$$\frac{(\mathbf{C}_i)^2}{(1-\beta\cos 2\phi)^{\frac{1}{2}}}=\{1+\tfrac{1}{32}\beta^2(j^2-6)\}\{1-\tfrac{1}{2}\beta(j-1)\cos 2\phi\} \quad . \quad . \quad . \quad . \quad (76).$$

This is also unity to the order at present needed.

Hence

$$\int p\left\{\frac{(\mathfrak{C}_i)^2}{(\mathbf{C}_i)^2}[(\mathfrak{P}_i)^2-(\text{P}_i)^2]d\sigma=\frac{4\pi}{2i+1}\text{M}(1-\beta)^{\frac{1}{2}}\{\tfrac{1}{2}\beta[j+1-(2i+1)]\right.$$
$$+\tfrac{1}{64}\beta^2[5j^2+14j+12-4(2i+1)(j+3)]\}$$
$$=\frac{4\pi\text{M}}{2i+1}\{\tfrac{1}{2}\beta(j+1)+\tfrac{1}{64}\beta^2(5j^2-2j-4)\}$$
$$-\pi\text{M}\{2\beta+\tfrac{1}{4}\beta^2(j-1)\} \quad . \quad . \quad . \quad (77).$$

§ 18. *Preparation for the Integrations when $s=1$ and 0.*

We have now to evaluate the three integrals

$$\left.\begin{aligned}\text{L}&=\int p(\mathfrak{S}_i^1\,\text{P}_i^1)^2\,d\sigma,\\ \text{M}&=\int p\left(\mathfrak{C}_i^1\,\frac{\text{P}_i^1}{\cos\theta}\right)^2\left(\frac{1+\beta}{1-\beta}-\sin^2\theta\right)d\sigma,\\ \text{N}&=\int p(\mathfrak{C}_i\,\text{P}_i)^2\,d\sigma\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (78),$$

and from these to determine three others when $\mathbf{S}$, $\mathbf{C}$ replace $\mathfrak{S}$, $\mathfrak{C}$.

We have

$$\frac{pd\sigma}{d\theta d\phi}=\left\{\frac{M(1-\beta)^{\frac{1}{2}}\left(\frac{1-\beta}{1+\beta}\right)^{\frac{1}{2}}}{(1-\frac{1-\beta}{1+\beta}\sin^2\theta)^{\frac{1}{2}}}\right\}\left\{\frac{\cos^2\theta+\frac{\beta(1-\cos 2\phi)}{1-\beta}}{(1-\beta\cos 2\phi)^{\frac{1}{2}}}\right\},$$

$$\frac{p\,(\frac{1+\beta}{1-\beta}-\sin^2\theta)\,d\sigma}{d\theta\,d\phi}=\left\{M(1-\beta)^{\frac{1}{2}}\left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}\left(1-\frac{1-\beta}{1+\beta}\sin^2\theta\right)^{\frac{1}{2}}\right\}\left\{\frac{\cos^2\theta+\frac{\beta(1-\cos 2\phi)}{1-\beta}}{(1-\beta\cos 2\phi)^{\frac{1}{2}}}\right\}.$$

It is the second factor which alone involves ϕ, and as I shall now first integrate with respect to ϕ, the first factor may be dropped for the moment, and the second factor multiplied by the squares of the cosine or sine functions. Since the integration is from $\phi = 2\pi$ to 0, those terms which vanish on integration may be dropped.

For brevity write

$$j = i(i+1),$$
$$j_0 = \tfrac{1}{32}(j^2 - 4j + 6),$$
$$j_1 = \tfrac{1}{256}(j^2 - 12j + 68).$$

Then we have seen in (73) and (75) that

$$\frac{(\mathfrak{S}_i^1)^2}{(1-\beta\cos 2\phi)^{\frac{1}{2}}} = \tfrac{1}{2}(1 - \tfrac{1}{4}\beta + \beta^2 j_1)\{1 - [1 + \tfrac{1}{8}\beta(j-4)]\cos 2\phi\},$$
$$\frac{(\mathfrak{C}_i^1)^2}{(1-\beta\cos 2\phi)^{\frac{1}{2}}} = \tfrac{1}{2}(1 + \tfrac{1}{4}\beta + \beta^2 j_1)\{1 + [1 - \tfrac{1}{8}\beta(j-4)]\cos 2\phi\},$$
$$\frac{(\mathfrak{C}_i)^2}{(1-\beta\cos 2\phi)^{\frac{1}{2}}} = (1 + \beta^2 j_0)\{1 - \tfrac{1}{2}\beta(j-1)\cos 2\phi]\}.$$

Therefore

$$\left.\begin{aligned}
&\int_0^{2\pi} \frac{\cos^2\theta + \frac{\beta(1-\cos 2\phi)}{1-\beta}}{(1-\beta\cos 2\phi)^{\frac{1}{2}}} \cdot (\mathfrak{S}_i^1)^2\, d\phi = \pi(1 - \tfrac{1}{4}\beta + \beta^2 j_1)\{\cos^2\theta + \tfrac{3}{2}\beta + \tfrac{1}{16}\beta^2(j+20)\},\\
&\int_0^{2\pi} \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ (\mathfrak{C}_i^1)^2\, d\phi = \pi(1 + \tfrac{1}{4}\beta + \beta^2 j_1)\{\cos^2\theta + \tfrac{1}{2}\beta + \tfrac{1}{16}\beta^2(j+4)\},\\
&\int_0^{2\pi} \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ (\mathfrak{C}_i)^2\, d\phi = 2\pi(1 + \beta^2 j_0)\{\cos^2\theta + \beta + \tfrac{1}{4}\beta^2(j+3)\}
\end{aligned}\right\} \quad . \quad (78).$$

Now pick out the parts of $pd\sigma$ and of these integrals (78) which are independent of θ, and write

$$\left.\begin{aligned}
F &= \pi M(1-\beta)^{\frac{1}{2}}\left(\frac{1-\beta}{1+\beta}\right)^{\frac{1}{2}}(1 - \tfrac{1}{4}\beta + \beta^2 j_1)\\
&= \pi M\{1 - \tfrac{7}{4}\beta + \tfrac{1}{256}\beta^2(j^2 - 12j + 388)\},\\
G &= \pi M(1-\beta)^{\frac{1}{2}}\left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}(1 + \tfrac{1}{4}\beta + \beta^2 j_1)\\
&= \pi M\{1 + \tfrac{3}{4}\beta + \tfrac{1}{256}\beta^2(j^2 - 12j + 68)\},\\
H &= 2\pi M(1-\beta)^{\frac{1}{2}}\left(\frac{1-\beta}{1+\beta}\right)^{\frac{1}{2}}(1 + \beta^2 j_0)\\
&= 2\pi M\{1 - \tfrac{3}{2}\beta + \tfrac{1}{32}\beta^2(j^2 - 4j + 34)\}
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (79).$$

Also write

$$\begin{aligned}
f &= \tfrac{3}{2}(1 + \tfrac{1}{24}\beta j + \tfrac{5}{6}\beta),\\
g &= \tfrac{1}{2}(1 + \tfrac{1}{8}\beta j + \tfrac{1}{2}\beta),\\
h &= 1 + \tfrac{1}{4}\beta j + \tfrac{3}{4}\beta,
\end{aligned} \quad . \quad . \quad . \quad (80).$$

$$\kappa'^2 = 1 - \kappa^2 = \frac{1-\beta}{1+\beta}, \text{ so that } \kappa^2 = 2\beta + 2\beta^2 + \ldots$$

Lastly, in accordance with the usual notation for elliptic integrals, write

$$\Delta^2 = 1 - \frac{1-\beta}{1+\beta}\sin^2\theta = 1 - \kappa'^2 \sin^2\theta \quad \ldots \ldots \quad (80).$$

Then we have

$$\left.\begin{aligned} \mathrm{L} &= \mathrm{F}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos^2\theta + \beta f}{\Delta} (\mathrm{P}_i^1)^2 d\theta \\ \mathrm{M} &= \mathrm{G}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (\cos^2\theta + \beta g).\Delta.(\mathrm{P}_i^1)^2 d\theta \\ \mathrm{N} &= \mathrm{H}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos^2\theta + \beta h}{\Delta} (\mathrm{P}_i)^2 d\theta \end{aligned}\right\} \quad \ldots \ldots \quad (81).$$

The next step is to express the squares of the P's in a series of powers of $\cos^2\theta$.

When $\mathrm{P}_i^s(\mu) = \dfrac{(1-\mu^2)^{\frac{1}{2}s}}{2^i . i!}\left(\dfrac{d}{d\mu}\right)^{i+s}(\mu^2-1)^i$, it is known that

$$\mathrm{P}_i\left(\mu\mu' + (1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}}\cos\phi\right) = \mathrm{P}_i(\mu)\,\mathrm{P}_i(\mu') + 2\sum_{s=1}^{s=i}\frac{i-s!}{i+s!}\mathrm{P}_i^s(\mu)\,\mathrm{P}_i^s(\mu')\cos s\phi.$$

By putting $\mu = \mu'$ we see that $2\dfrac{i-s!}{i+s!}(\mathrm{P}_i^s(\mu))^2$ is the coefficient of $\cos s\phi$ in the expansion of $\mathrm{P}_i(1-(1-\mu^2)\,2\sin^2\frac{1}{2}\phi)$. By TAYLOR'S theorem this last is equal to

$$\sum_{r=0}^{r=i}(-)^r\frac{(1-\mu^2)^r}{r!}(2\sin^2\tfrac{1}{2}\phi)^2\left(\frac{d^r}{d\mu^r}\mathrm{P}_i(\mu),\ \mu=1\right).$$

Now

$$\left(\frac{d}{d\mu}\right)^r \mathrm{P}_i = \frac{1}{2^i\, i!}\left(\frac{d}{d\mu}\right)^{i+r}(\mu^2-1)^i = \frac{1}{2^r . r!}\,\frac{i+r!}{i-r!}[1 + \text{terms involving powers of } \mu^2-1],$$

$$= \frac{1}{2^r . r!}\,\frac{i+r!}{i-r!}, \text{ when } \mu = 1.$$

Also

$$\sin^{2r}\tfrac{1}{2}\phi = \left(\frac{e^{\frac{1}{2}\phi\sqrt{-1}} - e^{-\frac{1}{2}\phi\sqrt{-1}}}{2\sqrt{-1}}\right)^{2r},$$

$$= \sum_{t=0}^{t=r}(-)^{r-t}\,\frac{2r!}{2^{2r}.\,2r-t!\,t!}\,e^{(r-t)\phi\sqrt{-1}}.$$

On putting $r - t = s$, we see that the coefficient of $\cos s\phi$ in $\sin^{2r}\frac{1}{2}\phi$ is

$$\frac{(-)^s}{2^{2r-1}}\cdot\frac{2r!}{r-s!\;r+s!}.$$

Hence we have*

$$2\frac{i-s\,!}{i+s\,!}(\mathrm{P}_i^s(\mu))^2 = 2\sum_{r=s}^{r=i}(-)^{r+s}\frac{2r\,!}{2^{2r}\,r-s\,!\,(r\,!)^2\,r+s\,!}\,\frac{i+r\,!}{i-r\,!}\cos^{2r}\theta \quad . \quad . \quad . \quad (82).$$

Now suppose

$$(\mathrm{P}_i^1)^2 = \sum_1^i \gamma_{2r-2}\cos^{2r}\theta,$$

$$(\mathrm{P}_i)^2 = \sum_0^i \alpha_{2r}\cos^{2r}\theta.$$

Then clearly

$$\gamma_{2r-2} = (-)^{r+1}\frac{2r\,!}{2^{2r}\,r-1\,!\,(r\,!)^2\,r+1\,!}\cdot\frac{i+r\,!}{i-r\,!}\cdot\frac{i+1\,!}{i-1\,!},$$

$$\alpha_{2r} = (-)^r\frac{2r\,!}{2^{2r}(r\,!)^4}\,\frac{i+r\,!}{i-r\,!} \quad . \quad . \quad . \quad . \quad . \quad (82).$$

Therefore

$$\frac{\mathrm{L}}{\mathrm{F}} = \sum_{r=1}^{r=i}\gamma_{2r-2}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{\cos^{2r+2}\theta+\beta f\cos^{2r}\theta}{\Delta}\,d\theta,$$

$$\frac{\mathrm{M}}{\mathrm{G}} = \sum_{r=1}^{r=i}\gamma_{2r-2}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}(\cos^{2r}\theta+\beta g\cos^{2r-2}\theta)\,\Delta d\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83).$$

$$\frac{\mathrm{N}}{\mathrm{G}} = \sum_{r=0}^{r=i}\alpha_{2r}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{\cos^{2r+2}\theta+\beta h\cos^{2r}\theta}{\Delta}\,d\theta$$

The evaluation of these integrals depends on two integrals only, namely, $\int\frac{\cos^{2n}\theta}{\Delta}\,d\theta$ and $\int\cos^{2n}\theta\,.\,\Delta d\theta$, and these will be considered in the next section.

§ 19. *Evaluation of the Integrals* $\int\frac{\cos^{2n}\theta}{\Delta}d\theta$ *and* $\int\cos^{2n}\theta\Delta d\theta$.

I will denote these integrals D and E respectively, and I propose to find their values in series proceeding by powers of κ^2.

The usual notation is adopted where $\Pi(x)$ is such a function that it is equal to $x\Pi(x-1)$; accordingly when x is a positive integer $\Pi(x) = x\,!$.

Since κ'^2 is less than unity

$$\frac{1}{\Delta} = \sum_0^\infty\frac{\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2r-1}{2}}{r\,!}\,\kappa'^{2r}\sin^{2r}\theta\,;$$

and since

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\cos^{2n}\theta\sin^{2r}\theta d\theta = \pi\,\frac{\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2r-1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2n-1}{2}}{n+r\,!},$$

$$\mathrm{D} = \int\frac{\cos^{2n}\theta}{\Delta}\,d\theta = \pi\,\frac{\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2n-1}{2}}{n\,!}\sum_0^\infty\frac{\left(\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2r-1}{2}\right)^2\kappa'^2}{(n+1)(n+2)\ldots(n+r)r\,!},$$

* Mr. Hobson kindly gave me this proof when I had shown him the series which I believed to hold true.

or, with the usual notation for hypergeometric series,

$$D=\pi\frac{2n\,!}{2^{2n}(n\,!)^2}F(\tfrac{1}{2},\tfrac{1}{2},n+1,\kappa'^2).$$

This series is of no service, since it proceeds by powers of κ'^2, which in our case is nearly unity. It is required then to transform the series into one proceeding by powers of κ^2.

It is known that, if $\kappa^2+\kappa'^2=1$,

$$F(a,b,c,\kappa'^2)=\frac{\Pi(c-a-b-1)\,\Pi(c-1)}{\Pi(c-a-1)\,\Pi(c-b-1)}F(a,b,1+a+b-c,\kappa^2)$$

$$+\kappa^{2(c-a-b)}\frac{\Pi(a+b-c-1)\,\Pi(c-1)}{\Pi(a-1)\,\Pi(b-1)}F(c-a,c-b,c-a-b+1,\kappa^2).*$$

If we apply this theorem with $a=b=\frac{1}{2}$, $c=n+1$, the first F becomes $F(\frac{1}{2},\frac{1}{2},1-n,\kappa^2)$, whose nth and all subsequent terms involve zero factors in the denominators. Also the coefficient of the second F involves $\Pi(-n-1)$, which has an infinite factor. Hence the formula leads to an indeterminate result. Let us therefore put $c=n+1+\epsilon$, and proceed to the limit when $\epsilon=0$.

We have then

$$D=\text{Limit}\,\pi\frac{2n\,!}{2^{2n}(n\,!)^2}\left\{\frac{\Pi(n-1+\epsilon)\,\Pi(n+\epsilon)}{[\Pi(n-\frac{1}{2}+\epsilon)]^2}F(\tfrac{1}{2},\tfrac{1}{2},1-n-\epsilon,\kappa^2)\right.$$

$$\left.+\kappa^{2(n+\epsilon)}\frac{\Pi(-n-1-\epsilon)\,\Pi(n+\epsilon)}{[\Pi(-\frac{1}{2})]^2}F(n+\tfrac{1}{2}+\epsilon,n+\tfrac{1}{2}+\epsilon,n+1+\epsilon,\kappa^2)\right\}.$$

Now $\Pi(\epsilon)=1+\epsilon\,\Pi'(0)$, $\qquad \Pi(-\frac{1}{2}+\epsilon)=\Pi(-\frac{1}{2})\left(1+\epsilon\frac{\Pi'(-\frac{1}{2})}{\Pi(-\frac{1}{2})}\right)$.

Therefore, when ϵ is very small,

$$\frac{\Pi(n-1+\epsilon)}{\Pi(n-1)}=1+\epsilon\left(\frac{1}{n-1}+\frac{1}{n-2}+\ldots+\tfrac{1}{2}+1\right)+\epsilon\,\Pi'(0)$$

$$=1+\epsilon\left(\Pi'(0)+\sum_1^n\frac{1}{t}-\frac{1}{n}\right)$$

$$\frac{\Pi(n+\epsilon)}{\Pi(n)}=1+\epsilon\left(\Pi'(0)+\sum_1^n\frac{1}{t}\right)$$

$$\frac{\Pi(n-\frac{1}{2}+\epsilon)}{\Pi(n-\frac{1}{2})}=1+2\epsilon\left(\frac{1}{2n-1}+\frac{1}{2n-3}+\ldots+\tfrac{1}{3}+1\right)+\epsilon\frac{\Pi'(-\frac{1}{2})}{\Pi(-\frac{1}{2})}$$

$$=1+\epsilon\left(\frac{\Pi'(-\frac{1}{2})}{\Pi(-\frac{1}{2})}+2\sum_1^n\frac{1}{2t-1}\right).$$

* I have to thank Mr. HOBSON for giving me this formula, and for showing me the procedure whereby it can be made effective.

Hence for the coefficient of the first series we have

$$\frac{\Pi(n-1+\epsilon)\,\Pi(n+\epsilon)}{[\Pi(n-\frac{1}{2}+\epsilon)]^2}=\frac{\Pi(n-1)\,\Pi(n)}{\Pi(n-\frac{1}{2})}\left\{1+\epsilon\left(2\Pi'(0)+2\sum_1^n\frac{1}{t}-\frac{1}{n}-2\frac{\Pi'(-\frac{1}{2})}{\Pi(-\frac{1}{2})}-4\sum_1^n\frac{1}{2t-1}\right)\right\}.$$

But $\Pi(-\frac{1}{2})=\pi^{\frac{1}{2}}$, $\quad \Pi'(0)-\dfrac{\Pi'(-\frac{1}{2})}{\Pi(-\frac{1}{2})}=\log_e 4$,*

$$[\Pi(n-\tfrac{1}{2})]^2=\pi\left(\frac{2n\,!}{2^{2n}\,n\,!}\right)^2,\qquad \Pi(n)\,\Pi(n-1)=n\,!\;n-1\,!$$

Therefore

$$\pi\,\frac{2n\,!}{2^{2n}(n\,!)^2}\,\frac{\Pi(n-1+\epsilon)\,\Pi(n+\epsilon)}{[\Pi(n-\frac{1}{2}+\epsilon)]^2}=2^{2n}\,\frac{n\,!\,n-1\,!}{2n\,!}\left\{1+\epsilon\left(2\log 4-\frac{1}{n}+2\sum_1^n\frac{1}{t}-4\sum_1^n\frac{1}{2t-1}\right)\right\}.$$

This is true from $n=\infty$ to 1, but in the case of $n=0$ we have

$$\Pi(-1+\epsilon)=\frac{1}{\epsilon}\Pi(\epsilon)=\frac{1}{\epsilon}+\Pi'(0),$$

so that in that case $\quad \pi\,\dfrac{\Pi(-1+\epsilon)\,\Pi(\epsilon)}{[\Pi(-\frac{1}{2}+\epsilon)]^2}=\dfrac{1}{\epsilon}+2\log 4.$

Now consider the coefficient of the second series.

We have $\quad \kappa^{2n+2\epsilon}=\kappa^{2n}(1+2\epsilon\log_e\kappa),$

and since $\quad \Pi(-x)\,\Pi(x-1)=\dfrac{\pi}{\sin\pi x}$

$$\Pi(-n-1-\epsilon)\,\Pi(n+\epsilon)=\frac{\pi}{\sin(n+1+\epsilon)\,\pi}=\frac{(-)^{n+1}}{\epsilon}\,\pi,\qquad [\Pi(-\tfrac{1}{2})]^2=\pi.$$

Therefore the coefficient of the second series is $(-)^{n+1}\dfrac{2n\,!}{2^{2n}(n\,!)^2}\,\dfrac{\kappa^{2n}}{\epsilon}\,(1+2\epsilon\log\kappa)$, and

$$\mathrm{D}=\frac{2^{2n}\,n\,!\,n-1\,!}{2n\,!}\left[1+\epsilon\left(2\log 4-\frac{1}{n}+2\sum_1^n\frac{1}{t}-4\sum_1^n\frac{1}{2t-1}\right)\right]F(\tfrac{1}{2},\tfrac{1}{2},1-n-\epsilon,\kappa^2)$$

$$+(-)^{n+1}\frac{2n!}{2^{2n}(n)!^2}\,\kappa^{2n}\,\frac{1}{\epsilon}\,(1+2\epsilon\log\kappa)\,F(n+\tfrac{1}{2}+\epsilon,n+\tfrac{1}{2}+\epsilon,n+1+\epsilon,\kappa^2).$$

The case of $n=0$ is an exception, for the coefficient of the first F has the part inside [ ] replaced by $\dfrac{1}{\epsilon}(1+2\epsilon\log 4)$.

* Proved by differentiating the known formula $\Pi(x-1)\,\Pi(x-\frac{1}{2})=\Pi(2x-1)\,.\,\dfrac{(4\pi)^{\frac{1}{2}}}{4^x}$, and putting $x=\frac{1}{2}$.

It remains to consider these two F series.

$$F(\tfrac{1}{2}, \tfrac{1}{2}, 1-n-\epsilon, \kappa^2) = \sum_0^\infty \frac{[1.3\ldots(2r-1)]^2 \kappa^{2r}}{2^{2r}(1-n-\epsilon)(2-n-\epsilon)\ldots(r-n-\epsilon).r!}$$

$$= \sum_0^{n-1} (-)^r \frac{[1.3\ldots(2r-1)]^2 \kappa^{2r}}{2^{2r}(n-1+\epsilon)(n-2+\epsilon)\ldots(n-r+\epsilon)r!}$$

$$+(-)^n \sum_n^\infty \frac{[1.3\ldots(2r-1)]^2 \kappa^{2r}}{2^{2r}(n-1+\epsilon)\ldots(1+\epsilon)\epsilon(1-\epsilon)\ldots(r-n-\epsilon).r!}$$

When $r<n$

$$\frac{1}{(n-1+\epsilon)(n-2+\epsilon)\ldots(n-r+\epsilon)} = \frac{1}{(n-1)(n-2)\ldots(n-r)}\left(1-\epsilon\sum_{n-r}^{n-1}\frac{1}{t}\right)$$

When $r>n$, put $r=n+s$, and

$$\frac{1}{(n-1+\epsilon)(n-2+\epsilon)\ldots(1+\epsilon)\epsilon(1-\epsilon)\ldots(s-\epsilon)} = \frac{1}{n-1!\,s!}\cdot\frac{1}{\epsilon}\left[1-\epsilon\sum_1^n\frac{1}{t}+\epsilon\sum_1^s\frac{1}{r}+\frac{\epsilon}{n}\right].$$

Also when $r=n+s$

$$\kappa^{2r}\frac{[1.3\ldots(2r-1)]^2}{2^{2r}.r!} = \frac{(2n!)^2}{2^{4n}(n!)^3}\kappa^{2n}\frac{[(2n+1)(2n+3)\ldots(2n+2s-1)]^2}{2^{2s}(n+1)(n+2)\ldots(n+s)}\kappa^{2s}.$$

Thus

$$F(\tfrac{1}{2}, \tfrac{1}{2}, 1-n-\epsilon, \kappa^2) = \sum_{r=0}^{r=n-1}(-)^r\frac{[1.3\ldots(2r-1)]^2}{2^{2r}(n-1)\ldots(n-r)r!}\left(1-\epsilon\sum_{n-r}^{n-1}\frac{1}{t}\right)\kappa^{2r}$$

$$+(-)^n\frac{(2n!)^2}{2^{4n}(n!)^4}n\kappa^{2n}\sum_{s=0}^{s=\infty}\frac{[(2n+1)\ldots(2n+2s-1)]^2}{2^{2s}(n+1)\ldots(n+s)s!}\frac{1}{\epsilon}\left(1-\epsilon\sum_1^n\frac{1}{t}+\frac{\epsilon}{n}+\epsilon\sum_1^s\frac{1}{t}\right)\kappa^{2s}.$$

It follows that we may write the first term of D as follows :—

$$(-)^n\frac{2n!}{2^{2n}(n!)^2}\frac{\kappa^{2n}}{\epsilon}F(n+\tfrac{1}{2}, n+\tfrac{1}{2}, n+1, \kappa^2) + \frac{2^{2n}n!\,n-1!}{2n!}\sum_0^{n-1}\frac{(-)^r[1.3\ldots(2r-1)]^2}{2^{2r}(n-1)\ldots(n-r).r!}\kappa^{2r}$$

$$+(-)^n\frac{2n!}{2^{2n}(n!)^2}\kappa^{2n}\sum_0^\infty\frac{[(2n+1)\ldots(2n+2s-1)]^2}{2^{2s}(n+1)\ldots(n+s).s!}\left[\sum_1^n\frac{1}{t}+\sum_1^s\frac{1}{t}-4\sum_1^n\frac{1}{2t-1}+2\log 4\right]\kappa^{2s}.$$

The first of these terms becomes infinite when $\epsilon=0$.

Turning to the second F we have

$$F(n+\tfrac{1}{2}+\epsilon, n+\tfrac{1}{2}+\epsilon, n+1+\epsilon, \kappa^2) = \sum_0^\infty\frac{[2n+1+2\epsilon)(2n+3+2\epsilon)\ldots(2n+2s-1+2\epsilon)]^2}{2^{2s}(n+1+\epsilon)(n+2+\epsilon)\ldots(n+s+\epsilon).s!}\kappa^{2s}$$

$$= \sum_0^\infty\frac{[(2n+1)(2n+3)\ldots(2n+2s-1)]^2}{2^{2s}(n+1)(n+2)\ldots(n+s).s!}\left[1+4\epsilon\sum_{n+1}^{n+}\frac{1}{2t-1}-\epsilon\sum_{n+}^{n+s}\frac{1}{t}\right]\kappa^{2s}.$$

Thus the second term of D is

$$(-)^{n+1}\frac{2n\,!}{2^{2n}(n!)^2}\frac{\kappa^{2n}}{\epsilon}F(n+\tfrac{1}{2}, n+\tfrac{1}{2}, n+1, \kappa^2)$$

$$+(-)^{n+1}\frac{2n\,!}{2^{2n}(n\,!)^2}\kappa^{2n}\sum_0^\infty\frac{[(2n+1)\dots(2n+2s-1)]^2}{2^{2s}(n+1)\dots(n+s)s\,!}\left[4\sum_{n+1}^{n+s}\frac{1}{2t-1}-\sum_{n+1}^{n+s}\frac{1}{t}+2\log\kappa\right]\kappa^{2s}.$$

The first term of this becomes infinite when $\epsilon=0$, but it is equal and opposite to the infinite term in the first part of D, and they annihilate one another.

Hence.

$$D=\frac{2^{2n}n\,!\;n-1\,!}{2n\,!}\sum_0^{n-1}\frac{(-)^r[1.3\dots(2r-1)]^2}{2^{2r}(n-1)\dots(n-r)r\,!}\kappa^{2r}$$

$$+(-)^n\frac{2n!}{2^{2n}(n!)^2}\kappa^{2n}\sum_0^\infty\frac{[(2n+1)\dots(2n+2s-1)]^2}{2^{2s}(n+1)\dots(n+s)\,.\,s\,!}\left[2\log\frac{4}{\kappa}+\sum_1^{n+}\frac{1}{t}+\sum_1^{s}\frac{1}{t}-4\sum_1^{n+s}\frac{1}{2t-1}\right]\kappa^{2s}.$$

On examining the case of $n=0$ we find that this formula also embraces it, provided we interpret $\sum_1^0$ as meaning zero.

The coefficient in the last term admits of some simplification, for

$$\sum_1^{n+}\frac{1}{t}+\sum_1^{s}\frac{1}{t}-4\sum_1^{n+s}\frac{1}{2t-1}=-\left[2\sum_1^{n+s}\frac{1}{t(2t-1)}+\sum_{t=1}^{t=n}\frac{1}{t+s}\right].$$

We thus conclude that D or

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{\cos^{2n}\theta}{\Delta}d\theta=\frac{2^{2n}n\,!\;n-1\,!}{2n\,!}\left[1-\frac{1^2}{2^2(n-1)1\,!}\kappa^2+\frac{1^2\,.\,3^2}{2^4(n-1)(n-2)2\,!}\kappa^4-\dots\text{to } n \text{ terms}\right]$$

$$+(-)^n\frac{2n!}{2^{2n}(n!)^2}\kappa^{2n}\Bigg[\left(2\log\frac{4}{\kappa}-\sum_1^n\frac{1}{t}-2\sum_1^n\frac{1}{t(2t-1)}\right)$$

$$+\frac{(2n+1)^2}{2^2(n+1)1\,!}\left(2\log\frac{4}{\kappa}-\sum_2^{n+1}\frac{1}{t}-2\sum_1^{n+1}\frac{1}{t(2t-1)}\right)\kappa^2$$

$$+\frac{(2n+1)^2(2n+3)^2}{2^4(n+1)(n+2)2\,!}\left(2\log\frac{4}{\kappa}-\sum_3^{n+2}\frac{1}{t}-2\sum_1^{n+2}\frac{1}{t(2t-1)}\right)\kappa^4+\dots\Bigg]\quad\quad(84).$$

The second integral E may be found as follows :—

$$E_n=\int\cos^{2n}\theta\,\Delta d\theta=\int\cos^{2n}\theta\,[\kappa^2+(1-\kappa^2)\cos^2\theta]\,\frac{d\theta}{\Delta},$$

$$=\kappa^2 D_n+(1-\kappa^2)\,D_{n+1}\quad\dots\dots\quad(85).$$

From this I find E or

$$\int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^{2n}\theta \,.\, \Delta d\theta = \frac{2^{2n+1}(n!)^2}{2n+1!} + \frac{2^{2n-1}\, n!\, n-1!}{2n+1!} \sum_0^{n-1} (-)^r \frac{1\,.\,3^2 \ldots (2r-1)^2\,(2r+1)}{2^{2r}(n-1)(n-2)\ldots(n-r)\,.\,r+1!}\,\kappa^{2r+2}$$

$$+ \frac{(-)^n\, 2n!}{2^{2n+1}\, n!\, n+1!}\,\kappa^{2n} \sum_1^{\infty} \frac{(2n+1)(2n+3)^2 \ldots (2n+2s-3)^2\,(2n+2s-1)}{2^{2s-2}(n+2)(n+3)\ldots(n+s)\,.\,s-1!}$$

$$\times \left[2\log\frac{4}{\kappa} + \sum_1^{n+s}\frac{1}{t} + \sum_1^{s-1}\frac{1}{t} - 4\sum_1^{n+s-1}\frac{1}{2t-1} - \frac{2}{2n+2s-1}\right]\kappa^{2s}.$$

This is applicable also to the case of $n = 0$, provided that $\sum\limits_0^{n-1}$ is interpreted as zero. In the particular case in hand I find, however, that it is shorter not to use this general formula, but to carry out the transformation (85) in the particular cases where the result is needed.

§ 20. *Reduction of preceding integrals; disappearance of logarithmic terms.*

In the application of the integrals of the last section, we are to put $\kappa^2 = 1 - \frac{1-\beta}{1+\beta}$, and only to develop as far as β^2.

Then to the proposed order $\kappa^2 = 2\beta(1-\beta)$, $\kappa^4 = 4\beta^2$.

Also
$$2\log\frac{4}{\kappa} = \log\frac{8}{\beta} + \log(1+\beta) = \log\frac{8}{\beta} + \beta - \tfrac{1}{2}\beta^2.$$

It will now facilitate future developments to adopt an abridged notation. I write then

$$f(n) = \frac{2^{2n}\, n!\, n-1!}{2n!},$$

and observe that $f(n+1) = \frac{2n}{2n+1} f(n)$, and $f(1) = 2$, $f(2) = \frac{4}{3}$.

Since κ^2 is of the first order in β, only the first series in the D integral (84) enters when n is greater than 2. In that case

$$D = f(n)\left[1 - \frac{\beta - \beta^2}{2(n-1)} + \frac{9\beta^2}{8(n-1)(n-2)}\right],$$

$$= f(n)\left[1 - \frac{\beta}{2(n-1)} + \frac{(4n+1)\beta^2}{8(n-1)(n-2)}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (86).$$

This result may be obtained very shortly without reference to the general formula; for when n is greater than 2

$$D = \left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos^{2n}\theta\, d\theta}{(\cos^2\theta + 2\beta + 2\beta^2)^{\frac{1}{2}}},$$

$$= \left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^{2n-1}\theta \left[1 - \frac{\beta + \beta^2}{\cos^2\theta} + \frac{3\beta^2}{2\cos^4\theta}\right] d\theta.$$

The integral of an odd power of $\cos\theta$ is easily determined, and it will be found that the result (86) is obtained. It is, however, clear that if n is not greater than 2 the development in powers of $\sec^2\theta$ is not legitimate.

When n is not greater than 2 the formula (84) of the last section is necessary, and we find

$$\left.\begin{aligned}\int\frac{\cos^4\theta}{\Delta}\,d\theta&=\tfrac{3}{2}\beta^2\log\frac{8}{\beta}+f(2)\left[1-\tfrac{1}{2}\beta-\frac{61}{16}\beta^2\right];\\ \int\frac{\cos^2\theta}{\Delta}\,d\theta&=-(\beta+\tfrac{5}{4}\beta^2)\log\frac{8}{\beta}+f(1)\left[1+\tfrac{3}{2}\beta+\frac{19}{16}\beta^2\right];\\ \int\frac{d\theta}{\Delta}&=(1+\tfrac{1}{2}\beta+\tfrac{1}{16}\beta^2)\log\frac{8}{\beta}-\tfrac{5}{16}\beta^2\end{aligned}\right\}\quad\ldots\quad(87).$$

We have now to find the second integral E, and this may be done more easily than by reference to the general formula of the last section.

We have

$$\mathrm{E}=\int\cos^{2n}\theta\Delta d\theta=(1-\kappa^2)\int\frac{\cos^{2n+2}\theta}{\Delta}\,d\theta+\kappa^2\int\frac{\cos^{2n}\theta}{\Delta}\,d\theta,$$

$$=(1-2\beta+2\beta^2)\int\frac{\cos^{2n+2}\theta}{\Delta}\,d\theta+2\beta\,(1-\beta)\int\frac{\cos^{2n}\theta}{\Delta}\,d\theta.$$

It will be observed that even when n is 2 the general formula (86) gives the D integral as far as the first power of β. Hence in finding E we may use that general formula except when $n=0,\ 1$.

Then since $f(n)=\dfrac{2n+1}{2n}f(n+1)$, when n is greater than 1,

$$\mathrm{E}=f(n+1)\left[(1-2\beta+2\beta^2)\left(1-\frac{\beta}{2n}+\frac{(4n+5)\,\beta^2}{8n\,(n-1)}\right)+\frac{2n+1}{n}\beta\,(1-\beta)\left(1-\frac{\beta}{2\,(n-1)}\right)\right],$$

$$=f(n+1)\left[1+\frac{\beta}{2n}-\frac{(4n-1)}{8n\,(n-1)}\beta^2\right]\quad\ldots\quad(88).$$

But when $n=1$,

$$\mathrm{E}=(1-2\beta+2\beta^2)\,.\,\tfrac{3}{2}\beta^2\log\frac{8}{\beta}-2\beta\,(1-\beta)\,.\,\beta(1+\tfrac{5}{4}\beta)\log\frac{8}{\beta}$$

$$+f(2)\left[(1-2\beta+2\beta^2)\,(1-\tfrac{1}{2}\beta-\tfrac{61}{16}\beta^2)+3\beta\,(1-\beta)\,(1+\tfrac{3}{2}\beta)\right],$$

$$=-\tfrac{1}{2}\beta^2\log\frac{8}{\beta}+f(2)\left[1+\tfrac{1}{2}\beta+\tfrac{11}{16}\beta^2\right]\quad\ldots\quad(89).$$

And when $n=0$,

$$\mathrm{E}=-(1-2\beta+2\beta^2)\beta\,(1+\tfrac{5}{4}\beta)\log\frac{8}{\beta}+2\beta\,(1-\beta)\,(1+\tfrac{1}{2}\beta)\log\frac{8}{\beta}$$

$$+f(1)\left[(1-2\beta+2\beta^2)\,(1+\tfrac{3}{2}\beta+\tfrac{19}{16}\beta^2)-2\beta\,(1-\beta)\,.\,\tfrac{5}{16}\beta^2\right],$$

$$=\beta\,(1-\tfrac{1}{4}\beta)\log\frac{8}{\beta}+f(1)\left[1-\tfrac{1}{2}\beta+\tfrac{3}{16}\beta^2\right]\quad\ldots\quad(90).$$

I now wish to show that, in the use to which these integrals are to be put, the logarthmic terms disappear.

The following is a table of these integrals collected from (87), (89), (90), in as far only as they involve logarithms :—

$$\int \frac{d\theta}{\Delta} = (1+\tfrac{1}{2}\beta+\tfrac{1}{16}\beta^2)\log\frac{8}{\beta}, \qquad \int \Delta d\theta = \beta(1-\tfrac{1}{4}\beta)\log\frac{8}{\beta},$$

$$\int \frac{\cos^2\theta}{\Delta}\, d\theta = -\beta(1+\tfrac{5}{4}\beta)\log\frac{8}{\beta}, \quad \int \cos^2\theta\,\Delta\, d\theta = -\tfrac{1}{2}\beta^2\log\frac{8}{\beta},$$

$$\int \frac{\cos^4\theta}{\Delta}\, d\theta = \tfrac{3}{2}\beta^2\log\frac{8}{\beta}.$$

Then the formulæ (83) for L, M, N, in so far only as is at present material, are

$$\frac{\mathrm{L}}{\mathrm{F}} = \int\left[\frac{\gamma_0}{\Delta}(\cos^4\theta+\beta f\cos^2\theta)+\frac{\gamma_2}{\Delta}\beta f\cos^4\theta\right]d\theta.$$

$$\frac{\mathrm{M}}{\mathrm{G}} = \int[\gamma_0\Delta\,(\cos^2\theta+\beta g)+\gamma_2\,\Delta\beta g\cos^2\theta]\,d\theta.$$

$$\frac{\mathrm{N}}{\mathrm{H}} = \int\left[\frac{\alpha_0}{\Delta}(\cos^2\theta+\beta h)+\frac{\alpha_2}{\Delta}(\cos^4\theta+\beta h\cos^2\theta)+\frac{\alpha_4}{\Delta}\beta h\cos^4\theta\right]d\theta.$$

On using the integrals and only retaining squares of β, we find

$$\frac{\mathrm{L}}{\mathrm{F}} = \beta^2\gamma_0\,(\tfrac{3}{2}-f)\log\frac{8}{\beta}.$$

$$\frac{\mathrm{M}}{\mathrm{G}} = \beta^2\gamma_0\,(g-\tfrac{1}{2})\log\frac{8}{\beta}.$$

$$\frac{\mathrm{N}}{\mathrm{H}} = \{\beta\alpha_0\,[-(1+\tfrac{5}{4}\beta)+h\,(1+\tfrac{1}{2}\beta)]+\beta^2\alpha_2(\tfrac{3}{2}-h)\}\log\frac{8}{\beta}.$$

But by definition of f and g in (80) and of the α's in (82), to the order zero of small quantities,

$$f=\tfrac{3}{2},\ g=\tfrac{1}{2},\ h=1+\tfrac{1}{4}\beta j+\tfrac{3}{4}\beta,\ \alpha_0=1,\ \alpha_2=-\tfrac{1}{2}i(i+1)=-\tfrac{1}{2}j.$$

Thus the logarithmic terms entirely disappear, and henceforth may be dropped.

Thus, *as far as material*, we have the following table of integrals :—

$$\left.\begin{array}{l}
\int\frac{d\theta}{\Delta}=-\tfrac{5}{16}\beta^2, \qquad \int\frac{\cos^2\theta}{\Delta}\,d\theta=f(1)[1+\tfrac{3}{2}\beta+\tfrac{19}{16}\beta^2].\\
\int\frac{\cos^4\theta}{\Delta}\,d\theta=f(2)[1-\tfrac{1}{2}\beta-\tfrac{61}{16}\beta^2].\\
\int\frac{\cos^{2n}\theta}{\Delta}\,d\theta=f(n)\left[1-\frac{\beta}{2(n-1)}+\frac{(4n+1)\beta^2}{8(n-1)(n-2)}\right],\ n>2.\\
\int\Delta\,d\theta=f(1)[1-\tfrac{1}{2}\beta+\tfrac{3}{16}\beta^2].\\
\int\cos^2\theta\Delta\,d\theta=f(2)[1+\tfrac{1}{2}\beta+\tfrac{11}{16}\beta^2].\\
\int\cos^{2n}\theta\Delta\,d\theta=f(n+1)\left[1+\frac{\beta}{2n}-\frac{(4n-1)\beta^2}{8n(n-1)}\right],\ n>1
\end{array}\right\} \quad . \quad . \quad . \quad (91).$$

Before using these for the determination of L, M, N, it is well to obtain one other result.

We have seen in (82) that

$$(\mathrm{P}^s)^2=\frac{i+s\,!}{i-s\,!}\sum_s^i(-)^{r+s}\frac{2r\,!}{2^{2r}r+s\,!\,(r\,!)^2r-s\,!}\cos^{2r}\theta.$$

Therefore $$\int_{-1}^{+1}(\mathrm{P}^s)^2d\mu=\frac{i+s\,!}{i-s\,!}\sum_s^i(-)^{r+s}f(r+1)\frac{2r\,!}{2^{2r}r+s\,!\,(r\,!)^2r-s\,!}.$$

But this integral is equal to $\dfrac{2}{2i+1}\,\dfrac{i+s\,!}{i-s\,!}$; therefore

$$\sum_s^i(-)^{r+s}f(r+1)\frac{2r\,!}{2^{2r}r+s\,!\,(r\,!)^2r-s\,!}=\frac{2}{2i+1}.$$

Putting $s=1$ and 0, and comparing with the values of α_{2r}, γ_{2r-2} in (82), we have

$$\left.\begin{array}{l}
\sum_0^i\alpha_{2r}f(r+1)=\dfrac{2}{2i+1}.\\
\sum_1^i\gamma_{2r-2}f(r+1)=\dfrac{2}{2i+1}\cdot\dfrac{i+1\,!}{i-1\,!}
\end{array}\right\} \quad . \quad . \quad . \quad (92).$$

§ 21. *Integrals of the squares of harmonics when $s=1$ and $s=0$.*

In (83) we have

$$\frac{\mathrm{L}}{\mathrm{F}}=\sum_1^i\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{\gamma_{2r-2}}{\Delta}(\cos^{2r+2}\theta+\beta f\cos^{2r}\theta)\,d\theta.$$

Therefore, noting that $f(r)=\dfrac{2r+1}{2r}f(r+1)$, and using the integrals (91),

$$\frac{\mathrm{L}}{\mathrm{F}}=\sum_2^i\gamma_{2r-2}f(r+1)\left[1-\frac{\beta}{2r}+\frac{(4r+5)\beta^2}{8r(r-1)}+\frac{2r+1}{2r}\beta\left(1-\frac{\beta}{2(r-1)}\right)f\right]$$
$$+\gamma_0f(2)[1-\tfrac{1}{2}\beta-\tfrac{61}{16}\beta^2+\tfrac{3}{2}\beta(1+\tfrac{3}{2}\beta)f].$$

Substituting for f (which the reader must not confuse with the functional f in use here) its value (80), the term of order zero is $\sum_1^i \gamma_{2r-2} f(r+1)$. By (92) this is equal to $\frac{2}{2i+1} \cdot \frac{i+1!}{i-1!}$.

The term of the first order in β is

$$\beta \sum_2^i \gamma_{2r-2} f(r+1) \left(-\frac{1}{2r} + \frac{3(2r+1)}{4r}\right) + \beta\gamma_0 f(2)\,(-\tfrac{1}{2}+\tfrac{3}{4}),$$

which may be reduced to the form

$$\beta \sum_1^i \gamma_{2r-2} f(r+1) + \tfrac{1}{2}\beta \sum_1^i \gamma_{2r-2} f(r), \text{ and is equal to } \frac{2\beta}{2i+1}\,\frac{i+1!}{i-1!} + \tfrac{1}{2}\beta \sum_1^i \gamma_{2r-2} f(r).$$

The term of the second order in β is

$$\beta^2 \sum_2^i \gamma_{2r-2} f(r+1) \left[\frac{4r+5}{8r(r-1)} + \frac{3(2r+1)}{4r}\left(-\frac{1}{2(r-1)} + \tfrac{1}{24}j + \tfrac{5}{6}\right)\right]$$
$$+ \beta^2\gamma_0 f(2)\,[-\tfrac{61}{16} + \tfrac{9}{4}(\tfrac{3}{2} + \tfrac{1}{24}j + \tfrac{5}{6})].$$

This may be reduced to the form

$$\beta^2 \left[\tfrac{1}{2}\sum_1^i \gamma_{2r-2} f(r+1) + \tfrac{1}{16}(j+12) \sum_1^i \gamma_{2r-2} f(r) - \tfrac{1}{4}\gamma_0\right],$$

of which the first term is $\tfrac{1}{2}\beta^2 \cdot \frac{2}{2i+1}\,\frac{i+1!}{i-1!}$.

Therefore

$$\frac{L}{F} = (1+\beta+\tfrac{1}{2}\beta^2)\frac{2}{2i+1}\,\frac{i+1!}{i-1!} + \tfrac{1}{2}\beta[1+\tfrac{1}{8}\beta(j+12)] \sum_1^i \gamma_{2r-2} f(r) - \tfrac{1}{4}\beta^2\gamma_0.$$

Now
$$\gamma_{2r-2} f(r) = (-)^{r+1}\,\frac{i+1\,!}{i-1\,!}\,\frac{1}{r+1\,!\,r\,!}\,\frac{i+r\,!}{i-r\,!},$$

and
$$\sum_1^i \frac{(-)^{r+1}}{r+1!\,r!}\,\frac{i+r!}{i-r!} = \frac{(i+1)i}{1\,!\,2} - \frac{(i+1)(i+2)\,i(i-1)}{2\,!\,2\,.\,3} + \ldots$$
$$= 1 - F(i+1, -i, 2, 1).$$

It is known that

$$F(a, b, c, 1) = \frac{\Pi(c-1)\Pi(c-a-b-1)^*}{\Pi(c-a-1)\Pi(c-b-1)} \quad \ldots \ldots \quad (93)$$

Then since $\Pi(-i)$ contains an infinite factor

$$F(i+1, -i, 2, 1) = \frac{\Pi(1)\,\Pi(0)}{\Pi(-i)\,\Pi(i+1)} = 0,$$

therefore
$$\sum_1^i \gamma_{2r-2} f(r) = \frac{i+1\,!}{i-1\,!} \quad \ldots \ldots \quad (93).$$

* I have again to thank Mr. HOBSON for this formula, which is due to GAUSS.

Also $$\gamma_0=\tfrac{1}{4}i^2(i+1)^2=\tfrac{1}{4}j\,\frac{i+1\,!}{i-1\,!}.$$

Hence $$\frac{\mathrm{L}}{\mathrm{F}}=\frac{2}{2i+1}\,\frac{i+1\,!}{i-1\,!}(1+\beta+\tfrac{1}{2}\beta^2)+\tfrac{1}{2}\beta\,\frac{i+1\,!}{i-1\,!}(1+\tfrac{3}{2}\beta).$$

Introducing the value of F as defined in (79), we have L or

$$\int p(\mathfrak{S}_i^1\mathrm{P}_i^1)^2d\sigma=\frac{2\pi\mathrm{M}}{2i+1}\cdot\frac{i+1\,!}{i-1\,!}[1-\tfrac{3}{4}\beta+\tfrac{1}{256}\beta^2(j^2-12j+68)]$$
$$+\pi\mathrm{M}\,\frac{i+1\,!}{i-1\,!}[\tfrac{1}{2}\beta-\tfrac{1}{8}\beta^2].$$

We have in (74) obtained $\int p(\mathfrak{S}_i^1)^2[(\mathfrak{P}_i^1)^2-(\mathrm{P}_i^1)^2]d\sigma$, and if it be added to our last result we see that the term which does not involve the factor $1/(2i+1)$ is annihilated, and

$$\int p(\mathfrak{P}_i^1\mathfrak{S}_i^1)^2d\sigma=\frac{2\pi\mathrm{M}}{2i+1}\,\frac{i+1\,!}{i-1\,!}\,[1+\tfrac{1}{8}\beta(j-2)+\tfrac{1}{384}\beta^2(j^2-26j+48)]\quad . \quad . \quad (94).$$

Now from (37) the square of the factor for converting $\mathfrak{S}^1$ into $\mathbf{S}^1$ is

$$\left[\frac{1}{\mathrm{D}_i^1}(\sin)\right]^2=1+\tfrac{1}{2}\beta+\tfrac{1}{32}\beta^2(j-8).$$

Therefore

$$\int p\,(\mathfrak{P}_i^1\,\mathbf{S}_i^1)^2\,d\sigma=\frac{2\pi\mathrm{M}}{2i+1}\cdot\frac{i+1\,!}{i-1\,!}\left[1+\tfrac{1}{8}\beta\,(j+2)+\tfrac{1}{384}\beta^2\,(j^2+10j-96)\right].\quad . \quad . \quad (94).$$

These are two of the required integrals.

Next we have from (83)

$$\frac{\mathrm{M}}{\mathrm{G}}=\sum_1^i\int\gamma_{2r-2}\,(\cos^{2r}\theta+\beta g\cos^{2r-2}\theta)\,\Delta d\theta.$$

Noting as before that $f(r)=\dfrac{2r+1}{2r}f(r+1)$, and using the integrals (91),

$$\frac{\mathrm{M}}{\mathrm{G}}=\sum_2^i\gamma_{2r-2}f(r+1)\left[1+\frac{\beta}{2r}-\frac{(4r-1)\beta^2}{8(r-1)r}+\frac{2r+1}{2r}\beta g\left(1+\frac{\beta}{2(r-1)}\right)\right]$$
$$+\gamma_0 f(2)\left[1+\tfrac{1}{2}\beta+\tfrac{11}{16}\beta^2+\tfrac{3}{2}\beta g\,(1-\tfrac{1}{2}\beta)\right].$$

Substituting for g its value from (80), I find the term of order zero to be $\sum_1^i\gamma_{2r-2}f(r+1)$, or $\dfrac{2}{2i+1}\cdot\dfrac{i+1\,!}{i-1\,!}$.

The term of the first order is $\beta\sum_2^i\gamma_{2r-2}f(r+1)\left[\dfrac{1}{2r}+\dfrac{2r+1}{4r}\right]+\beta\gamma_0 f(2)\,(\tfrac{3}{4}+\tfrac{1}{2})$.

This may be reduced to the form $-\beta\sum_1^i\gamma_{2r-2}f(r+1)+\frac{3}{2}\beta\sum_1^i\gamma_{2r-2}f(r)$; which by (92) and (93) becomes $-\frac{2\beta}{2i+1}\frac{i+1\,!}{i+1\,!}+\frac{3}{2}\beta\frac{i+1\,!}{i-1\,!}$.

The term of the second order is

$$\beta^2\sum_2^i\gamma_{2r-2}f(r+1)\left[-\frac{(4r-1)}{8r(r-1)}+\frac{2r+1}{4r}\left(\frac{1}{2(r-1)}+\frac{1}{8}j+\frac{1}{2}\right)\right]+\beta^2\gamma_0 f(2)\left(\frac{11}{16}+\frac{3}{32}j\right).$$

This is reducible to

$$\tfrac{1}{2}\beta^2\sum_1^i\gamma_{2r-2}f(r+1)+\tfrac{1}{16}\beta^2(j-4)\sum_1^i\gamma_{2r-2}f(r)+\tfrac{3}{4}\beta^2\gamma_0;\ \text{which becomes}$$

$$\tfrac{1}{2}\beta^2\frac{2}{2i+1}\cdot\frac{i+1\,!}{i-1\,!}+\tfrac{1}{16}\beta^2(j-4)\frac{i+1\,!}{i-1\,!}+\tfrac{3}{16}\beta^2\frac{i+1\,!}{i-1\,!}j.$$

Therefore

$$\frac{M}{G}=\frac{2}{2i+1}\frac{i+1\,!}{i-1\,!}(1-\beta+\tfrac{1}{2}\beta^2)+\frac{i+1\,!}{i-1\,!}[\tfrac{3}{2}\beta+\tfrac{1}{4}\beta^2(j-1)].$$

Introducing for G its value (79), we find M or

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}p\left(\mathfrak{C}_i^1\frac{P_i^1}{\cos\theta}\right)^2\left(\frac{1+\beta}{1-\beta}-\sin^2\theta\right)d\sigma=\frac{2\pi M}{2i+1}\frac{i+1\,!}{i-1\,!}\left[1-\tfrac{1}{4}\beta+\tfrac{1}{256}\beta^2(j^2-12j+4)\right]$$
$$+\pi M\frac{i+1\,!}{i-1\,!}\left[\tfrac{3}{2}\beta+\tfrac{1}{8}\beta^2(2j+7)\right].$$

But in (74) we have $\int p(\mathfrak{C}_i^1)^2\left[(\mathsf{P}_i^1)^2-\left(P_i^1\sqrt{\frac{\frac{1+\beta}{1-\beta}-\sin^2\theta}{\cos^2\theta}}\right)^2\right]d\sigma$. If this be added to the result just found the term which has not $1/(2i+1)$ as a factor is annihilated, and

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}p(\mathsf{P}_i^1\mathfrak{C}_i^1)^2d\sigma=\frac{2\pi M}{2i+1}\frac{i+1\,!}{i-1\,!}\left[1+\tfrac{1}{8}\beta(3j+10)+\tfrac{1}{384}\beta^2(29j^2+134j+384)\right].\quad(95).$$

Now from (37) the square of the factor for converting $\mathfrak{C}^1$ into C^1 is

$$\left[\frac{1}{D_i^1}(\cos)\right]^2=1-\tfrac{1}{2}\beta+\tfrac{1}{32}\beta^2(j-8).$$

Therefore

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}p(\mathsf{P}_i^1\mathsf{C}_i^1)^2d\sigma=\frac{2\pi M}{2i+1}\frac{i+1\,!}{i-1\,!}[1+\tfrac{3}{8}\beta(j+2)+\tfrac{1}{384}\beta^2(29j^2+74j+48)]\quad.\quad.\quad(95).$$

These last two complete the solution for $s=1$.

Next we have from (83)

$$\frac{\mathrm{N}}{\mathrm{H}}=\sum_0^i\int\frac{\alpha_{2r}}{\Delta}(\cos^{2r+2}\theta+\beta h\cos^{2r}\theta)\,d\theta.$$

Proceeding as before,

$$\frac{\mathrm{N}}{\mathrm{H}}=\sum_2^i\alpha_{2r}f(r+1)\left[1-\frac{\beta}{2r}+\frac{4r+5}{8r(r-1)}\beta^2+\frac{2r+1}{2r}\beta h\left(1-\frac{\beta}{2(r-1)}\right)\right]$$
$$+\alpha_2 f(2)\left[1-\tfrac{1}{2}\beta-\tfrac{61}{16}\beta^2+\tfrac{3}{2}\beta h(1+\tfrac{3}{2}\beta)\right]$$
$$+\alpha_0 f(1)\left[1+\tfrac{3}{2}\beta+\tfrac{19}{16}\beta^2\right]+\alpha_0\beta h(-\tfrac{5}{16}\beta^2).$$

Substituting for h its value (80), we find that the term of order zero is $\sum_0^i\alpha_{2r}f(r+1)$, and by (92) this is equal to $\frac{2}{2i+1}$.

The term of the first order is

$$\beta\sum_2^i\alpha_{2r}f(r+1)\left[-\frac{1}{2r}+\frac{2r+1}{2r}\right]+\tfrac{4}{3}\beta\alpha_2+3\beta\alpha_0,$$

which may be written in the form

$$\beta\sum_0^i\alpha_{2r}f(r+1)+\beta\alpha_0,\text{ and is equal to }\frac{2\beta}{2i+1}+\beta.$$

The term of the second order is

$$\beta^2\sum_2^i\alpha_{2r}f(r+1)\left[\frac{4r+5}{8r(r-1)}-\frac{2r+1}{4r(r-1)}+\frac{2r+1}{8r}(j+3)\right]+\beta^2\alpha_2(\tfrac{1}{2}j-\tfrac{7}{12})+\tfrac{19}{8}\beta^2\alpha_0,$$

which is equal to

$$\tfrac{3}{8}\beta^2\sum_2^i\alpha_{2r}\frac{f(r+1)}{r(r-1)}+\tfrac{1}{4}\beta^2(j+3)\sum_2^i\alpha_{2r}f(r)+\beta^2\alpha_2(\tfrac{1}{2}j-\tfrac{7}{12})+\tfrac{19}{8}\beta^2\alpha_0.$$

Now $$\tfrac{3}{8}\frac{f(r+1)}{r(r-1)}=\tfrac{1}{2}f(r+1)-\tfrac{3}{4}f(r)+\tfrac{1}{4}\frac{r}{r-1}f(r).$$

Hence the term may be written

$$\tfrac{1}{2}\beta^2\sum_2^i\alpha_{2r}f(r+1)+\tfrac{1}{4}\beta^2\sum_1^{i-1}\alpha_{2r+2}\frac{r+1}{r}f(r+1)+\tfrac{1}{4}\beta^2 j\sum_2^i\alpha_{2r}f(r)+\beta^2\alpha_2(\tfrac{1}{2}j-\tfrac{7}{12})+\tfrac{19}{8}\beta^2\alpha_0.$$

But $$\alpha_{2r}f(r)=(-)^r\frac{1}{(r!)^2 r}\frac{i+r!}{i-r!}.$$

And $$\frac{r+1}{r}\alpha_{2r+2}f(r+1)=-(-)^r\frac{1}{(r!)^2(r+1)^2 r}\left[i(i+1)-r(r+1)\right]\frac{i+r!}{i-r!}$$
$$=\left[-\frac{(-)^r}{(r!)^2 r}\cdot\frac{j}{(r+1)^2}+\frac{(-)^r}{(r!)^2(r+1)}\right]\frac{i+r!}{i-r!}.$$

In the preceding formula the sum of this last function had limits $i-1$ to 1, but as we now see that it vanishes when $r=i$, the upper limit may be changed to i.

It follows that the terms of the second order are

$$\tfrac{1}{2}\beta^2\sum_0^i \alpha_{2r} f(r+1) - \tfrac{1}{2}\beta^2\alpha_2 f(2) - \tfrac{1}{2}\beta^2\alpha_0 f(1).$$

$$+\tfrac{1}{4}\beta^2\sum_1^i\left[-\frac{(-)^r}{(r\,!)^2 r}\cdot\frac{j}{(r+1)^2}+\frac{(-)^r}{(r\,!)^2(r+1)}\right]\frac{i+r\,!}{i-r\,!}+\tfrac{1}{4}\beta^2 j\sum_1^i\frac{(-)^r}{r(r!)^2}\frac{i+r\,!}{i-r\,!}$$

$$-\tfrac{1}{4}\beta^2 j\alpha_2 f(1)+\beta^2\alpha_2(\tfrac{1}{2}j-\tfrac{7}{12})+\tfrac{19}{8}\beta^2\alpha_0.$$

The term in α_2 in this expression will be found to be $-\frac{5}{4}\alpha_2$. That in α_0 will be found to be $\frac{11}{8}\alpha_0$. Then since $\alpha_2=-\frac{1}{2}j$, $\alpha_0=1$, these terms are together $\frac{1}{8}\beta^2(5j+11)$.

The whole may then be written

$$\tfrac{1}{2}\beta^2\sum_0^i\alpha_{2r}f(r+1)+\tfrac{1}{4}\beta^2(j+1)\sum_1^i\frac{(-)^r}{r\,!\,r+1\,!}\frac{i+r\,!}{i-r\,!}+\tfrac{1}{4}\beta^2 j\sum_1^i\frac{(-)^r}{(r+1\,!)^2}\frac{i+r\,!}{i-r\,!}+\tfrac{1}{8}\beta^2(5j+11).$$

Now

$$\sum_1^i\frac{(-)^r}{r\,!\,r+1\,!}\frac{i+r\,!}{i-r\,!}=-\frac{(i+1)i}{1\,!\,2}+\frac{(i+1)(i+2)i(i-1)}{2\,!\,2\,.\,3}-\ldots=F(i+1,\,-i,\,2,\,1)-1=-1$$

$$\sum_1^i\frac{(-)^r}{(r+1)^2}\frac{i+r\,!}{i-r\,!}=-\frac{i(i+1)}{2\,!\,2\,!}+\frac{(i+1)(i+2)\,i(i-1)}{3\,!\,3\,!}-\ldots$$

$$=-\frac{1}{i(i+1)}\left[\frac{i(i+1)(-i-1)(-i)}{2\,!\,1\,.\,2}+\frac{i(i+1)(i+2)(-i-1)(-i)(-i+1)}{3\,!\,1\,.\,2\,.\,3}+\ldots\right]$$

$$=-\frac{1}{i(i+1)}\left\{F(i,-i-1,\,1,\,1)-1+i(i+1)\right\}$$

$$=-\frac{1}{j}F(i,-i-1,\,1,\,1)+\frac{1}{j}-1$$

$$=\ \frac{1}{j}-1.$$

The last result follows from the fact that in accordance with (93) the sum of the hypergeometric series has an infinite factor in the denominator, and vanishes.

Then since by (92) $\sum_0^i\alpha_{2r}f(r+1)=\dfrac{2}{2i+1}$, the terms of the second order are found to be

$$\tfrac{1}{2}\beta^2\cdot\frac{2}{2i+1}+\tfrac{1}{8}\beta^2 j+\tfrac{11}{8}\beta^2.$$

Hence, collecting terms,

$$\frac{\mathrm{N}}{\mathrm{H}}=(1+\beta+\tfrac{1}{2}\beta^2)\,\frac{2}{2i+1}+\beta+\tfrac{1}{8}\beta^2\,(j+11).$$

Substituting for H its value (79), we have N or

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}p\,(\mathfrak{C}_i\,\mathrm{P}_i)^2\,d\sigma=\frac{4\pi\mathrm{M}}{2i+1}\,[1-\tfrac{1}{2}\beta+\tfrac{1}{32}\beta^2(j^2-4j+2)]+2\pi\mathrm{M}\,[\beta+\tfrac{1}{8}\,\beta^2\,(j-1)].$$

But we have already found in (77) the value of $\int p\,(\mathfrak{C}_i)^2\,[(\mathfrak{P}_i)^2-(\mathrm{P}_i)^2]\,d\sigma$, and on adding it to the last result the term independent of $1/(2i+1)$ disappears, and we have

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} p\,(\mathfrak{P}_i\,\mathfrak{C}_i)^2\,d\sigma=\frac{4\pi\mathrm{M}}{2i+1}\left[1+\tfrac{1}{2}\beta j+\tfrac{1}{64}\beta^2(7j^2-10j)\right] \quad . \quad . \quad . \quad . \quad (96).$$

The square of the factor whereby $\mathfrak{C}_i$ is converted into $\mathbf{C}_i$ was found in (38), namely,

$$\frac{1}{(\mathrm{D}_i)^2}=1+\tfrac{1}{8}\beta^2(j-3).$$

Hence
$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} p\,(\mathfrak{P}_i\,\mathbf{C}_i)^2\,d\sigma=\frac{4\pi\mathrm{M}}{2i+1}\left[1+\tfrac{1}{2}\beta j+\tfrac{1}{64}\beta^2(7j^2-2j-24)\right] \quad . \quad . \quad . \quad (96).$$

These are the last of the required integrals.

§ 22. *Table of Integrals of squares of harmonics.**

In this section the results obtained in (71), (72), (94), (95), and (96) are collected.

* After having completed the evaluation of all these integrals, I found that they may be evaluated very shortly by means of the factors $\mathfrak{E}$ and $\mathbf{E}$ of (48), § 10.

I find that for *all* values of s (writing the eight forms in a single formula),

$$\int p\left\{\begin{matrix}(\mathfrak{P}_i{}^s)^2\\(\mathbf{P}_i{}^s)^2\end{matrix}\times\begin{matrix}(\mathfrak{C}_i{}^s)^2 \text{ or } (\mathbf{C}_i{}^s)^2\\(\mathfrak{S}_i{}^s)^2 \text{ or } (\mathbf{S}_i{}^s)^2\end{matrix}\right. d\sigma=\frac{4\pi\mathrm{M}}{2i+1}(1-\beta)^{\frac{1}{2}}\,.\left\{\begin{matrix}\mathfrak{E}_i{}^s\\\mathbf{E}_i{}^s\end{matrix}\times\left\{\text{const. part of }\frac{(\mathfrak{C}_i{}^s \text{ or } \mathbf{C}_i{}^s \text{ or } \mathfrak{S}_i{}^s \text{ or } \mathbf{S}_i{}^s)^2}{(1-\beta\cos 2\phi)^{\frac{1}{2}}}\right\}\right..$$

I leave the reader to verify that this is so.

Unfortunately I have hitherto been unable to prove the truth of this except by the laborious method in the text. I do not therefore know whether the result remains true for higher degrees of approximation, although I suspect it does so. If it should be true, it would be very easy to compute the integrals when higher powers of β are included.

It may be worth mentioning that the variables are separable in the integrals. Thus, when $\mathfrak{P}_i{}^s\,\mathfrak{C}_i{}^s$ denotes any one of the eight forms,

$$\frac{1}{\mathrm{M}(1-\beta)^{\frac{1}{2}}}\int p\,[\mathfrak{P}_i{}^s\,\mathfrak{C}_i{}^s]^2\,d\sigma=\frac{1}{\sqrt{1+\beta}}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{(\mathfrak{P}_i{}^s)^2\,d\theta}{\Delta}\int_0^{2\pi}(1-\beta\cos 2\phi)^{\frac{1}{2}}(\mathfrak{C}_i{}^s)^2\,d\phi$$
$$-\sqrt{\frac{1-\beta}{1+\beta}}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{\sin^2\theta\,(\mathfrak{P}_i{}^s)^2}{\Delta}\,d\theta\int_0^{2\pi}\frac{(\mathfrak{C}_i{}^s)^2}{(1-\beta\cos 2\phi)^{\frac{1}{2}}}\,d\phi.$$

The ϕ integrals present no difficulty, but with regard to the others we are met by the impossibility of expanding in powers of $\sec^2\theta$ for the lower orders. It would be a great step in the right direction, if it could be proved that all the terms which do not involve the factor $\dfrac{1}{2i+1}$ necessarily vanish.

It may be well to remind the reader that $M = k^3 \nu (\nu^2 - 1)^{\frac{1}{2}} \left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}$,

$$\Sigma = \frac{i(i+1)}{s^2-1},$$

$$\Upsilon = \frac{(i-1)i(i+1)(i+2)}{s^2-4},$$

$$j = i(i+1).$$

First when $s > 2$.

Types $\left\{\begin{matrix}\text{EEC}\\\text{OOS}\end{matrix}\right.$ $\int p \left(\mathfrak{P}_i^s(\mu) \left\{\begin{matrix}\mathfrak{C}_i^s\\\mathfrak{S}_i^s\end{matrix}\right.(\phi)\right)^2 d\sigma$

$$= \frac{2\pi M}{2i+1} \frac{i+s!}{i-s!} \{1 - \tfrac{1}{2}\beta\Sigma + \tfrac{1}{16}\beta^2[2\Sigma^2 + 3\Sigma - s^2(2\Sigma-1) + \Upsilon]\}.$$

Types $\left\{\begin{matrix}\text{OEC}\\\text{EOS}\end{matrix}\right.$ $\int p \left(\mathfrak{P}_i^s(\mu) \left\{\begin{matrix}\mathsf{C}_i^s\\\mathsf{S}_i^s\end{matrix}\right.(\phi)\right)^2 d\sigma$

$$= \frac{2\pi M}{2i+1} \frac{i+s!}{i-s!} \{1 - \tfrac{1}{2}\beta\Sigma + \tfrac{1}{16}\beta^2[2\Sigma^2 + \Sigma - 6 - s^2(2\Sigma-1) + \Upsilon]\}.$$

Types $\left\{\begin{matrix}\text{OOC}\\\text{EES}\end{matrix}\right.$ $\int p \left(\mathsf{P}_i^s(\mu) \left\{\begin{matrix}\mathfrak{C}_i^s\\\mathfrak{S}_i^s\end{matrix}\right.(\phi)\right)^2 d\sigma$

$$= \frac{2\pi M}{2i+1} \frac{i+s!}{i-s!} \{1 + \tfrac{1}{2}\beta(\Sigma+2) + \tfrac{1}{16}\beta^2[11\Sigma + 10 + s^2(2\Sigma^2 - 2\Sigma + 1) - \Upsilon]\}.$$

Types $\left\{\begin{matrix}\text{EOC}\\\text{OES}\end{matrix}\right.$ $\int p \left(\mathsf{P}_i^s(\mu) \left\{\begin{matrix}\mathsf{C}_i^s\\\mathsf{S}_i^s\end{matrix}\right.(\phi)\right)^2 d\sigma$

$$= \frac{2\pi M}{2i+1} \frac{i+s!}{i-s!} \{1 + \tfrac{1}{2}\beta(\Sigma+2) + \tfrac{1}{16}\beta^2[9\Sigma + 4 + s^2(2\Sigma^2 - 2\Sigma + 1) - \Upsilon]\}.$$

Secondly, when $s=2$, $\Sigma = \frac{1}{3}i(i+1)$.

Type EEC $\int p(\mathfrak{P}_i^2(\mu)\mathfrak{C}_i^2(\phi))^2 d\sigma = \frac{2\pi M}{2i+1}\frac{i+2!}{i-2!}\{1 - \frac{1}{2}\beta\Sigma + \frac{1}{256}\beta^2(95\Sigma^2 - 98\Sigma + 72)\}$.

Type OEC $\int p(\mathfrak{P}_i^2(\mu)\mathsf{C}_i^2(\phi))^2 d\sigma = \frac{2\pi M}{2i+1}\frac{i+2!}{i-2!}\{1 - \frac{1}{2}\beta\Sigma + \frac{1}{256}\beta^2(95\Sigma^2 - 178\Sigma - 40)\}$.

Type EES $\int p(\mathsf{P}_i^2(\mu)\mathfrak{S}_i^2(\phi))^2 d\sigma$

$$= \frac{2\pi M}{2i+1}\frac{i+2!}{i-2!}\{1 + \tfrac{1}{2}\beta(\Sigma+2) + \tfrac{1}{256}\beta^2(29\Sigma^2 + 90\Sigma + 216)\}.$$

Type OES $\int p(\mathsf{P}_i^2(\mu)\mathsf{S}_i^2(\phi))^2 d\sigma$

$$= \frac{2\pi M}{2i+1}\frac{i+2!}{i-2!}\{1 + \tfrac{1}{2}\beta(\Sigma+2) + \tfrac{1}{256}\beta^2(29\Sigma^2 + 106\Sigma + 136)\}.$$

Thirdly, when $s=1$, Σ is infinite and we must use $j = i(i+1)$.

Type OOS $\int p(\mathfrak{P}_i^1(\mu)\mathfrak{S}_i^1(\phi))^2 d\sigma = \frac{2\pi M}{2i+1}\frac{i+1!}{i-1!}\{1 + \frac{1}{8}\beta(j-2) + \frac{1}{384}\beta^2(j^2 - 26j + 48)\}$.

Type EOS $\int p\,(\mathfrak{P}_i^1(\mu)\,\mathsf{S}_i^1(\phi))^2\,d\sigma = \frac{2\pi M}{2i+1}\frac{i+1\,!}{i-1\,!}\{1+\frac{1}{8}\beta\,(j+2)+\frac{1}{384}\beta^2(j^2+10j-96)\}.$

Type OOC $\int p\,(\mathbf{P}_i^1(\mu)\,\mathfrak{C}_i^1(\phi))^2\,d\sigma$

$$= \frac{2\pi M}{2i+1}\frac{i+1\,!}{i-1\,!}\{1+\frac{1}{8}\beta\,(3j+10)+\frac{1}{384}\beta^2(29j^2+134j+384)\}.$$

Type EOC $\int p\,(\mathbf{P}_i^1(\mu)\,\mathbf{C}_i^1(\phi))^2 d\sigma = \frac{2\pi M}{2i+1}\frac{i+1\,!}{i-1\,!}\{1+\frac{3}{8}\beta(j+2)+\frac{1}{384}\beta^2(29j^2+74j+48)\}.$

Lastly, when $s=0$; $\Sigma=-i\,(i+1)=-j$. There are only two types—

Type EEC $\int p\,(\mathfrak{P}_i\,(\mu)\,\mathfrak{C}_i\,(\phi))^2\,d\sigma = \frac{4\pi M}{2i+1}\,[1+\frac{1}{2}\beta j+\frac{1}{64}\beta^2\,(7j^2-10j)].$

Type OEC $\int p\,(\mathfrak{P}_i(\mu)\,\mathbf{C}_i\,(\phi))^2\,d\sigma = \frac{4\pi M}{2i+1}\,[1+\frac{1}{2}\beta j+\frac{1}{64}\beta^2\,(7j^2-2j-24].$

PART III.

Summary.

The symmetrical form in which Lamé presented the three functions whose product is a solid ellipsoidal harmonic is such as to render purely analytical investigations both elegant and convenient. But it seemed to me that facility for computation might be gained by the surrender of symmetry, and I have acted on this idea in the preceding paper.

Spheroidal analysis has been successfully employed where the ellipsoid is one of revolution, and it therefore seemed advisable to make that method the point of departure for the treatment of ellipsoids with three unequal axes. In spheroidal harmonics we start with a fundamental prolate ellipsoid of revolution, with imaginary semi-axes $k\sqrt{-1}$, $k\sqrt{-1}$, 0. The position of a point is then defined by three co-ordinates; the first of these, ν, is such that its reciprocal is the eccentricity of a meridional section of an ellipsoid confocal with the fundamental ellipsoid and passing through the point. Since that eccentricity diminishes as we recede from the origin, ν plays the part of a reciprocal to the radius vector. The second co-ordinate, μ, is the cosine of the auxiliary angle in the meridional ellipse measured from the axis of symmetry. It therefore plays the part of sine of latitude. The third co-ordinate is simply the longitude ϕ. The three co-ordinates may then be described as the radial, latitudinal, and longitudinal co-ordinates. The parameter k defines the absolute scale on which the figure is drawn.

It is equally possible to start with a fundamental oblate ellipsoid with real axes k, k, 0. We should then take the first co-ordinate, ζ, as such that $\zeta^2 = -\nu^2$. All that follows would then be equally applicable; but, in order not to complicate the statement by continual reference to alternative forms, I shall adhere to the first form as a standard.

In this paper a closely parallel notation is adopted for the ellipsoid of three unequal axes. The squares of semi-axes of the fundamental ellipsoid are taken to be $-k^2\frac{1+\beta}{1-\beta}$, $-k^2$, 0, and the three co-ordinates are still ν, μ, ϕ. Although their geometrical meanings are now by no means so simple, they may still be described as radial, latitudinal, and longitudinal co-ordinates. As before, we might equally well start with a fundamental ellipsoid whose squares of semi-axes are $k^2\frac{1+\beta}{1-\beta}$, k^2, 0, and replace ν^2 by ζ^2, where $\zeta^2 = -\nu^2$. All possible ellipsoids are comprised in either of these types by making β vary from zero to infinity. But it is shown in § 2 that, by a proper choice of type, all possible ellipsoids are comprised in a range of β from zero to one-third. When β is zero we have the spheroids for which harmonic analysis already exists; and when $\beta = \frac{1}{3}$ the ellipsoid is such that the mean axis is the square root of mean square of the extreme axes. The harmonic analysis for this class of ellipsoid has not been yet worked out, but the method of this paper would render it possible to do so. We may then regard β as essentially less than $\frac{1}{3}$, and may conveniently make developments in powers of β.

In spheroidal analysis, for space internal to an ellipsoid ν_0, two of the three functions are the same P-functions that occur in spherical analysis; one P being a function of ν, the other of μ. The third function is a cosine or sine of a multiple of the longitude ϕ. In external space the P-function of ν is replaced by a Q-function, being a solution of the differential equation of the second kind.

The like is true in ellipsoidal analysis, and we have P- and Q-functions of ν for internal and external space, a P-function of μ, and a cosine- or sine-function of ϕ. I will now for a time set aside the Q-functions and consider them later.

There are eight cases to consider (§ 4); these are determined by the evenness or oddness of the degree i and of the order s of the harmonic, and by the alternative of whether they correspond with a cosine- or sine-function of ϕ. I indicate these eight types by the initials E, O, C, or S—for example, EOS means the type in which i is even, s is odd, and that there is association with a sine-function.

It appears that the new P-functions fall into two forms. The first form, which I write $\mathfrak{P}_i^s$, is found to be expressible in a finite series in terms of the $P_i^{s\pm 2k}$, where the P's are the ordinary functions of spherical analysis. The terms in this series are arranged in powers of β, so that the coefficient of $P_i^{s\pm 2k}$ has β^k as part of its coefficient. The second form, which I write P_i^s, is such that $\sqrt{\frac{\nu^2-1}{\nu^2-\frac{1+\beta}{1-\beta}}}\,\mathsf{P}_i^s(\nu)$ or $\sqrt{\frac{1-\mu^2}{\frac{1+\beta}{1-\beta}-\mu^2}}\,\mathsf{P}_i^s(\mu)$ is expressible by a series of the same kind as that for $\mathfrak{P}_i^s$. Amongst the eight types four involve $\mathfrak{P}$-functions and four P-functions; and if for given s a $\mathfrak{P}_i^s$-function is associated with a cosine-function, the corresponding P_i^s is associated with a sine-function, and *vice versâ*.

Lastly, a $\mathfrak{P}$-function of ν is always associated with a $\mathfrak{P}$-function of μ; and the like is true of the P's.

Again, the cosine- and sine-functions fall into two forms. In the first form s and i are either both even or both odd, and the function, which I write $\mathfrak{C}_i^s$ or $\mathfrak{S}_s^i$, is expressed by a series of terms consisting of a coefficient multiplied by β^k cos or sin $(s \pm 2k)\phi$. In the second form s and i differ as to evenness and oddness, and the function, written C_i^s or S_i^s, is expressed by a similar series multiplied by $(1 - \beta \cos 2\phi)^{\frac{1}{2}}$.

The combination of the two forms of P-function with the four forms of cosine- and sine-function gives the eight types of solid harmonic.

Corresponding to the two forms of P-function there are two forms of Q-function, such that $\mathfrak{Q}_i^s$ and $\mathsf{Q}_i^s \sqrt{\frac{\nu^2-1}{\nu^2-\frac{1+\beta}{1-\beta}}}$ are expansible in a series of ordinary Q-functions; but whereas the series for $\mathfrak{P}_i^s$ and P_i^s are terminable, because P_i^s vanishes when s is greater than i, this is not the case with the Q-functions. In fact the series for the Q-functions begins with Q_i or Q_i^1, and the order of the Q's increases by two at a time up to s when we have the principal or central term; it then goes on increasing up to $s = i$ or $i - 1$, and on to infinity.

In spherical and spheroidal analysis the differential equation satisfied by P_i^s involves the integer s, whereby the order is specified. So here also the differential equations, satisfied by $\mathfrak{P}_i^s$ or P_i^s and by $\mathfrak{C}_i^s$, $\mathfrak{S}_i^s$, or C_i^s, S_i^s, involve a constant, but it is no longer an integer. It seemed convenient to assume $s^2 - \beta\sigma$ as the form for this constant, where s is the known integer specifying the order of harmonic, and σ remains to be determined from the differential equations.

When the assumed forms for the P-function and for the cosine- and sine-functions are substituted in the differential equations, it is found (§ 6) that, in order to satisfy the equations, $\beta\sigma$ must be equal to the difference between two finite-continued fractions, each of which involves $\beta\sigma$. We thus have an equation for $\beta\sigma$, and the required root is that which vanishes when β vanishes.

For the harmonics of degrees 0, 1, 2, 3, and for all orders, σ may be found rigorously in algebraic form, but for higher degrees the equation can only be solved approximately, unless β should have a definite numerical value.

When $\beta\sigma$ has been determined, either rigorously or approximately, the successive coefficients of the series are determinable in such a way that the ratio of each coefficient to the preceding one is expressed by a continued fraction, which is, in fact, a portion of one of the two fractions involved in the equation for $\beta\sigma$.

Throughout the rest of the paper the greater part of the work is carried out with approximate forms, and, although it would be easy to attain to greater accuracy, I have thought it sufficient, in the first instance, to stop at β^2. With this limitation the coefficients of the series assume simple forms (§ 8), and we have thus definite, if approximate, expressions for all the functions which can occur in ellipsoidal analysis.

In rigorous expressions, $\mathfrak{P}_i^s$ and P_i^s are essentially different from one another, but in approximate forms, when s is greater than a certain integer dependent on the degree of approximation, the two are the same thing in different shapes, except as to a constant factor. I have, therefore, in § 9 determined up to squares of β the factors whereby P_i^s is convertible into $\mathfrak{P}_i^s$, and C_i^s or S_i^s into $\mathfrak{C}_i^s$ or $\mathfrak{S}_i^s$. With the degree of approximation adopted there is no factor for converting the P's when $s = 3, 2, 1$. Similarly, down to $s = 3$ inclusive, the same factor serves for converting C_i^s into $\mathfrak{C}_i^s$ and S_i^s into $\mathfrak{S}_i^s$. But for $s = 2, 1, 0$ one form is needed for changing C into $\mathfrak{C}$, and another for changing S into $\mathfrak{S}$. It may be well to note that there is no sine-function when s is zero.

The use of these factors does much to facilitate the laborious reductions involved in the whole investigation.

It is well known that the Q-functions are expressible in terms of the P-functions by means of a definite integral. Hence $\mathfrak{Q}_i^s$ and Q_i^s must have a second form, which can only differ from the other by a constant factor. The factors connecting the two forms are determined in § 10.

The second part of the paper is devoted to applications of the harmonic method. In § 11 the perpendicular from the centre on to the tangent plane to an ellipsoid ν_0, and the area of an element of surface of the ellipsoid, are found in terms of the co-ordinates μ, ϕ, and the constant ν_0.

It is easy to form a function, continuous at the surface ν_0, which shall be a solid harmonic both for external and for internal space. POISSON's equation then enables us to determine the surface density of which this continuous function is the potential, and it is found to be a surface harmonic of μ, ϕ multiplied by the perpendicular on to the tangent plane. This application of POISSON's equation involves the use of the Q-function in its integral form. Accordingly, if the serial form for the Q-function is adopted as a standard, the expression for the potential of a layer of surface density involves the use of the factor for conversion between the two forms of Q-function.

This result may obviously be employed to determine the potential of an harmonic deformation of a solid ellipsoid.

The potential of the solid ellipsoid itself may be found by the consideration that it is externally equal to that of a focaloid shell of the same mass. It appears that in order to express the equivalent surface density in surface harmonics, it is only necessary to express the reciprocal of the square of the perpendicular on the tangent plane in that form. This result is attained by expressing x^2, y^2, z^2 in surface harmonics. When this done, an application of the preceding theorem enables us to write down the external potential of the solid ellipsoid at once. In § 12 the external potential of the solid ellipsoid is expressed rigorously in terms of solid harmonics of degrees zero and 2.

Since x^2, y^2, z^2 have been found in surface harmonics, we can also write down a rotation-potential about any one of the three axes in the same form.

The internal potential of a solid ellipsoid does not lend itself well to elliptic co-ordinates, but expressions for it are given in § 12.

If it be desired to express any arbitrary function of μ, ϕ in surface harmonics, it is necessary to know the integrals, over the surface of the ellipsoid, of the squares of the several surface harmonics, each multiplied by the perpendicular on to the tangent plane. The rest of the paper is devoted to the evaluation of these integrals. No attempt is made to carry the developments beyond β^2, although the methods employed would render it possible to do so.

When s is greater than unity, it appears that it is legitimate to develop the function to be integrated in powers of $\frac{1}{1-\mu^2}$; and when this is done, the integration, although laborious, does not present any great difficulty.

But when s is either 1 or 0, the method of development breaks down, because it would give rise to infinite elements in the integrals at the poles where μ^2 is unity. However, portions of the integrals in these cases can still be found by the former method of development. As to the residues which cannot be so treated, it appears that they depend on integrals of the forms

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos^{2u}\theta\, d\theta}{(1-\kappa'^2\sin^2\theta)^{\frac{1}{2}}} \text{ and } \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^{2u}\theta\,(1-\kappa'^2\sin^2\theta)^{\frac{1}{2}}\, d\theta,$$

where κ'^2 is nearly equal to unity.

Development of the square-roots in powers of κ'^2 is useless on account of the slow convergence, and it is required to find series which proceed by powers of κ^2, where $\kappa^2 = 1 - \kappa'^2$.

By a somewhat difficult investigation, in respect to which I owe my special thanks to Mr. HOBSON, the needed series are found (§ 19).

It appears that portions of the two integrals involve logarithms which become infinite when κ vanishes. Since, in the application of these integrals, the vanishing of κ implies the vanishing of β, we appear to be met by a difficulty. It is known that in spheroidal analysis no such terms appear, and we may feel confident that they cannot really exist in ellipsoidal analysis. In § 20 it is proved that the logarithmic terms do as a fact disappear. The residues of the integrals in the cases $s = 1$, 0 are thus found, and added to the previous portions to form the complete results.

The second part of the paper ends (§ 22) with a list of the integrals of the squares of the surface harmonics for all values of s, as far as the squares of β.

Finally, an appendix below contains a table of all the functions as far as $i = 5$, $s = 5$. It is probable that for the higher values of s the results would only be applicable when β is very small.

APPENDIX.

Table of the P- and Q-Functions.

$i=0$ (EEC)

$$\left\{\begin{matrix}\mathfrak{P}_0\\ \mathfrak{Q}_0\end{matrix}\right.= \left\{\begin{matrix}P_0\\ Q_0\end{matrix}\right.+\tfrac{1}{4}\beta\left\{\begin{matrix}0\\ Q_0^2\end{matrix}\right.+\tfrac{1}{128}\beta^2\left\{\begin{matrix}0\\ Q_0^4\end{matrix}\right. .$$

$i=1$ (OEC)

$$\left\{\begin{matrix}\mathfrak{P}_1\\ \mathfrak{Q}_1\end{matrix}\right.= \left\{\begin{matrix}P_1\\ Q_1\end{matrix}\right.+\tfrac{1}{4}\beta\left\{\begin{matrix}0\\ Q_1^2\end{matrix}\right.+\tfrac{1}{128}\beta^2\left\{\begin{matrix}0\\ Q_1^4\end{matrix}\right.$$

(OOC)

$$\left\{\begin{matrix}\mathbf{P}_1^1\\ \mathbf{Q}_1^1\end{matrix}\right.=\Omega\left[\left\{\begin{matrix}P_1^1\\ Q_1^1\end{matrix}\right.+\tfrac{3}{16}\beta(1+\tfrac{1}{8}\beta)\left\{\begin{matrix}0\\ Q_1^3\end{matrix}\right.+\tfrac{5}{768}\beta^2\left\{\begin{matrix}0\\ Q_1^5\end{matrix}\right.\right]$$

(OOS)

$$\left\{\begin{matrix}\mathfrak{P}_1^1\\ \mathfrak{Q}_1^1\end{matrix}\right.= \left\{\begin{matrix}P_1^1\\ Q_1^1\end{matrix}\right.+\tfrac{1}{16}\beta(1-\tfrac{1}{8}\beta)\left\{\begin{matrix}0\\ Q_1^3\end{matrix}\right.+\tfrac{1}{768}\beta^2)\left\{\begin{matrix}0\\ Q_1^5\end{matrix}\right. .$$

$i=2$ (EEC)

$$\left\{\begin{matrix}\mathfrak{P}_2\\ \mathfrak{Q}_2\end{matrix}\right.= \left\{\begin{matrix}P_2\\ Q_2\end{matrix}\right.+\tfrac{1}{4}\beta\left\{\begin{matrix}P_2^2\\ Q_2^2\end{matrix}\right.+\tfrac{1}{128}\beta^2\left\{\begin{matrix}0\\ Q_2^4\end{matrix}\right.$$

(EOC)

$$\left\{\begin{matrix}\mathbf{P}_2^1\\ \mathbf{Q}_2^1\end{matrix}\right.=\Omega\left[\left\{\begin{matrix}P_2^1\\ Q_2^1\end{matrix}\right.+\tfrac{3}{16}\beta(1+\tfrac{3}{8}\beta)\left\{\begin{matrix}0\\ Q_2^3\end{matrix}\right.+\tfrac{5}{768}\beta^2\left\{\begin{matrix}0\\ Q_2^5\end{matrix}\right.\right]$$

(EOS)

$$\left\{\begin{matrix}\mathfrak{P}_2^1\\ \mathfrak{Q}_2^1\end{matrix}\right.= \left\{\begin{matrix}P_2^1\\ Q_2^1\end{matrix}\right.+\tfrac{1}{16}\beta(1-\tfrac{3}{8}\beta)\left\{\begin{matrix}0\\ Q_2^3\end{matrix}\right.+\tfrac{1}{768}\beta^2\left\{\begin{matrix}0\\ Q_2^5\end{matrix}\right.$$

(EEC)

$$\left\{\begin{matrix}\mathfrak{P}_2^2\\ \mathfrak{Q}_2^2\end{matrix}\right.= -3\beta\left\{\begin{matrix}P_2\\ Q_2\end{matrix}\right.+\left\{\begin{matrix}P_2^2\\ Q_2^2\end{matrix}\right.+\tfrac{1}{24}\beta\left\{\begin{matrix}0\\ Q_2^4\end{matrix}\right.+\tfrac{1}{1536}\beta^2\left\{\begin{matrix}0\\ Q_2^6\end{matrix}\right.$$

(EES)

$$\left\{\begin{matrix}\mathbf{P}_2^2\\ \mathbf{Q}_2^2\end{matrix}\right.=\Omega\left[\qquad\qquad \left\{\begin{matrix}P_2^2\\ Q_2^2\end{matrix}\right.+\tfrac{1}{12}\beta\left\{\begin{matrix}0\\ Q_2^4\end{matrix}\right.+\tfrac{1}{512}\beta^2\left\{\begin{matrix}0\\ Q_2^6\end{matrix}\right.\right] .$$

$i=3$ (OEC)

$$\left\{\begin{matrix}\mathfrak{P}_3\\ \mathfrak{Q}_3\end{matrix}\right.= \left\{\begin{matrix}P_3\\ Q_3\end{matrix}\right.+\tfrac{1}{4}\beta\left\{\begin{matrix}P_3^2\\ Q_3^2\end{matrix}\right.+\tfrac{1}{128}\beta^2\left\{\begin{matrix}0\\ Q_3^4\end{matrix}\right.$$

(OOC)

$$\left\{\begin{matrix}\mathbf{P}_3^1\\ \mathbf{Q}_3^1\end{matrix}\right.=\Omega\left[\left\{\begin{matrix}P_3^1\\ Q_3^1\end{matrix}\right.+\tfrac{3}{16}\beta(1+\tfrac{3}{4}\beta)\left\{\begin{matrix}P_3^3\\ Q_3^3\end{matrix}\right.+\tfrac{5}{768}\beta^2\left\{\begin{matrix}0\\ Q_3^5\end{matrix}\right.\right]$$

(OOS)

$$\left\{\begin{matrix}\mathfrak{P}_3^1\\ \mathfrak{Q}_3^1\end{matrix}\right.= \left\{\begin{matrix}P_3^1\\ Q_3^1\end{matrix}\right.+\tfrac{1}{16}\beta(1-\tfrac{3}{4}\beta)\left\{\begin{matrix}P_3^3\\ Q_3^3\end{matrix}\right.+\tfrac{1}{768}\beta^2\left\{\begin{matrix}0\\ Q_3^5\end{matrix}\right.$$

(OEC)

$$\left\{\begin{matrix}\mathfrak{P}_3^2\\ \mathfrak{Q}_3^2\end{matrix}\right.= -15\beta\left\{\begin{matrix}P_3\\ Q_3\end{matrix}\right.+\left\{\begin{matrix}P_3^2\\ Q_3^2\end{matrix}\right.+\tfrac{1}{24}\beta\left\{\begin{matrix}0\\ Q_3^4\end{matrix}\right.+\tfrac{1}{1536}\beta^2\left\{\begin{matrix}0\\ Q_3^6\end{matrix}\right.$$

(OES)

$$\left\{\begin{matrix}\mathbf{P}_3^2\\ \mathbf{Q}_3^2\end{matrix}\right.=\Omega\left[\qquad\qquad \left\{\begin{matrix}P_3^2\\ Q_3^2\end{matrix}\right.+\tfrac{1}{12}\beta\left\{\begin{matrix}0\\ Q_3^4\end{matrix}\right.+\tfrac{1}{512}\beta^2\left\{\begin{matrix}0\\ Q_3^6\end{matrix}\right.\right]$$

(OOC)

$$\left\{\begin{matrix}\mathbf{P}_3^3\\ \mathbf{Q}_3^3\end{matrix}\right.=\Omega\left[-\tfrac{5}{4}\beta(1+\tfrac{3}{4}\beta)\left\{\begin{matrix}P_3^1\\ Q_3^1\end{matrix}\right.+\left\{\begin{matrix}P_3^3\\ Q_3^3\end{matrix}\right.+\tfrac{5}{96}\beta\left\{\begin{matrix}0\\ Q_3^5\end{matrix}\right.+\tfrac{7}{7680}\beta^2\left\{\begin{matrix}0\\ Q_3^7\end{matrix}\right.\right]$$

(OOS)

$$\left\{\begin{matrix}\mathfrak{P}_3^3\\ \mathfrak{Q}_3^3\end{matrix}\right.= -\tfrac{15}{4}\beta(1-\tfrac{3}{4}\beta)\left\{\begin{matrix}P_3^1\\ Q_3^1\end{matrix}\right.+\left\{\begin{matrix}P_3^3\\ Q_3^3\end{matrix}\right.+\tfrac{1}{32}\beta\left\{\begin{matrix}0\\ Q_3^5\end{matrix}\right.+\tfrac{1}{2560}\beta^2\left\{\begin{matrix}0\\ Q_3^7\end{matrix}\right.$$

$i=4$ (EEC)
$$\left\{\begin{matrix}\mathfrak{P}_4\\ \mathfrak{Q}_4\end{matrix}\right.= \left\{\begin{matrix}P_4\\ Q_4\end{matrix}\right.+\tfrac{1}{4}\beta\left\{\begin{matrix}P_4^2\\ Q_4^2\end{matrix}\right.+\tfrac{1}{128}\beta^2\left\{\begin{matrix}P_4^4\\ Q_4^4\end{matrix}\right.$$

(EOC)
$$\left\{\begin{matrix}\mathsf{P}_4^1\\ \mathsf{Q}_4^1\end{matrix}\right.=\Omega\left[\left\{\begin{matrix}P_4^1\\ Q_4^1\end{matrix}\right.+\tfrac{3}{16}\beta(1+\tfrac{5}{4}\beta)\left\{\begin{matrix}P_4^3\\ Q_4^3\end{matrix}\right.+\tfrac{5}{768}\beta^2\left\{\begin{matrix}0\\ Q_4^5\end{matrix}\right.\right]$$

(EOS)
$$\left\{\begin{matrix}\mathfrak{P}_4^1\\ \mathfrak{Q}_4^1\end{matrix}\right.= \left\{\begin{matrix}P_4^1\\ Q_4^1\end{matrix}\right.+\tfrac{1}{16}\beta(1-\tfrac{5}{4}\beta)\left\{\begin{matrix}P_4^3\\ Q_4^3\end{matrix}\right.+\tfrac{1}{768}\beta^2\left\{\begin{matrix}0\\ Q_4^5\end{matrix}\right.$$

(EEC)
$$\left\{\begin{matrix}\mathfrak{P}_4^2\\ \mathfrak{Q}_4^2\end{matrix}\right.= -45\beta\left\{\begin{matrix}P_4\\ Q_4\end{matrix}\right.+\left\{\begin{matrix}P_4^2\\ Q_4^2\end{matrix}\right.+\tfrac{1}{24}\beta\left\{\begin{matrix}P_4^4\\ Q_4^4\end{matrix}\right.+\tfrac{1}{1536}\beta^2\left\{\begin{matrix}0\\ Q_4^6\end{matrix}\right.$$

(EES)
$$\left\{\begin{matrix}\mathsf{P}_4^2\\ \mathsf{Q}_4^2\end{matrix}\right.=\Omega\left[\qquad\left\{\begin{matrix}P_4^2\\ Q_4^2\end{matrix}\right.+\tfrac{1}{12}\beta\left\{\begin{matrix}P_4^4\\ Q_4^4\end{matrix}\right.+\tfrac{1}{512}\beta^2\left\{\begin{matrix}0\\ Q_4^6\end{matrix}\right.\right]$$

(EOC)
$$\left\{\begin{matrix}\mathsf{P}_4^3\\ \mathsf{Q}_4^3\end{matrix}\right.=\Omega\left[-\tfrac{21}{4}\beta(1+\tfrac{5}{4}\beta)\left\{\begin{matrix}P_4^1\\ Q_4^1\end{matrix}\right.+\begin{matrix}P_4^3\\ Q_4^3\end{matrix}+\tfrac{5}{96}\beta\left\{\begin{matrix}0\\ Q_4^5\end{matrix}\right.+\tfrac{7}{7680}\beta^2\left\{\begin{matrix}0\\ Q_4^7\end{matrix}\right.\right]$$

(EOS)
$$\left\{\begin{matrix}\mathfrak{P}_4^3\\ \mathfrak{Q}_4^3\end{matrix}\right.= -\tfrac{63}{4}(1-\tfrac{5}{4}\beta)\left\{\begin{matrix}P_4^1\\ Q_4^1\end{matrix}\right.+\left\{\begin{matrix}P_4^3\\ Q_4^3\end{matrix}\right.+\tfrac{1}{32}\beta\left\{\begin{matrix}0\\ Q_4^5\end{matrix}\right.+\tfrac{1}{2560}\beta^2\left\{\begin{matrix}0\\ Q_4^7\end{matrix}\right.$$

(EEC)
$$\left\{\begin{matrix}\mathfrak{P}_4^4\\ \mathfrak{Q}_4^4\end{matrix}\right.=\tfrac{105}{2}\beta^2\left\{\begin{matrix}P_4\\ Q_4\end{matrix}\right.-\tfrac{14}{3}\beta\left\{\begin{matrix}P_4^2\\ Q_4^2\end{matrix}\right.+\left\{\begin{matrix}P_4^4\\ Q_4^4\end{matrix}\right.+\tfrac{1}{40}\beta\left\{\begin{matrix}0\\ Q_4^6\end{matrix}\right.+\tfrac{1}{3840}\beta^2\left\{\begin{matrix}0\\ Q_4^8\end{matrix}\right.$$

(EES)
$$\left\{\begin{matrix}\mathsf{P}_4^4\\ \mathsf{Q}_4^4\end{matrix}\right.=\Omega\left[\qquad-\tfrac{7}{3}\beta\left\{\begin{matrix}P_4^2\\ Q_4^2\end{matrix}\right.+\left\{\begin{matrix}P_4^4\\ Q_4^4\end{matrix}\right.+\tfrac{3}{80}\beta\left\{\begin{matrix}0\\ Q_4^6\end{matrix}\right.+\tfrac{1}{1920}\beta^2\left\{\begin{matrix}0\\ Q_4^8\end{matrix}\right.\right].$$

$i=5$ (OEC)
$$\left\{\begin{matrix}\mathfrak{P}_5\\ \mathfrak{Q}_5\end{matrix}\right.= \left\{\begin{matrix}P_5\\ Q_5\end{matrix}\right.+\tfrac{1}{4}\beta\left\{\begin{matrix}P_5^2\\ Q_5^2\end{matrix}\right.+\tfrac{1}{128}\beta^2\left\{\begin{matrix}P_5^4\\ Q_5^4\end{matrix}\right]$$

(OOC)
$$\left\{\begin{matrix}\mathsf{P}_5^1\\ \mathsf{Q}_5^1\end{matrix}\right.=\Omega\left[\left\{\begin{matrix}P_5^1\\ Q_5^1\end{matrix}\right.+\tfrac{3}{16}\beta(1+\tfrac{15}{8}\beta)\left\{\begin{matrix}P_5^3\\ Q_5^3\end{matrix}\right.+\tfrac{5}{768}\beta^2\left\{\begin{matrix}P_5^5\\ Q_5^5\end{matrix}\right.\right]$$

(OOS)
$$\left\{\begin{matrix}\mathfrak{P}_5^1\\ \mathfrak{Q}_5^1\end{matrix}\right.= \left\{\begin{matrix}P_5^1\\ Q_5^1\end{matrix}\right.+\tfrac{1}{16}\beta(1-\tfrac{15}{8}\beta)\left\{\begin{matrix}P_5^3\\ Q_5^3\end{matrix}\right.+\tfrac{1}{768}\beta^2\left\{\begin{matrix}P_5^5\\ Q_5^5\end{matrix}\right.$$

(OEC)
$$\left\{\begin{matrix}\mathfrak{P}_5^2\\ \mathfrak{Q}_5^2\end{matrix}\right.= -105\beta\left\{\begin{matrix}P_5\\ Q_5\end{matrix}\right.+\left\{\begin{matrix}P_5^2\\ Q_5^2\end{matrix}\right.+\tfrac{1}{24}\beta\left\{\begin{matrix}P_5^4\\ Q_5^4\end{matrix}\right.+\tfrac{1}{1536}\beta^2\left\{\begin{matrix}0\\ Q_5^6\end{matrix}\right.$$

(OES)
$$\left\{\begin{matrix}\mathsf{P}_5^2\\ \mathsf{Q}_5^2\end{matrix}\right.=\Omega\left[\qquad\left\{\begin{matrix}P_5^2\\ Q_5^2\end{matrix}\right.+\tfrac{1}{12}\beta\left\{\begin{matrix}P_5^4\\ Q_5^4\end{matrix}\right.+\tfrac{1}{512}\beta^2\left\{\begin{matrix}0\\ Q_5^6\end{matrix}\right.\right]$$

(OOC)
$$\left\{\begin{matrix}\mathsf{P}_5^3\\ \mathsf{Q}_5^3\end{matrix}\right.=\Omega\left[-14\beta\,(1+\tfrac{15}{8}\beta)\left\{\begin{matrix}P_5^1\\ Q_5^1\end{matrix}\right.+\left\{\begin{matrix}P_5^3\\ Q_5^3\end{matrix}\right.+\tfrac{5}{96}\beta\left\{\begin{matrix}P_5^5\\ Q_5^5\end{matrix}\right.+\tfrac{7}{7680}\beta^2\left\{\begin{matrix}0\\ Q_5^7\end{matrix}\right.\right]$$

(OOS)
$$\left\{\begin{matrix}\mathfrak{P}_5^3\\ \mathfrak{Q}_5^3\end{matrix}\right.= -42\beta\,(1-\tfrac{15}{8}\beta)\left\{\begin{matrix}P_5^1\\ Q_5^1\end{matrix}\right.+\left\{\begin{matrix}P_5^3\\ Q_5^3\end{matrix}\right.+\tfrac{1}{32}\beta\left\{\begin{matrix}P_5^5\\ Q_5^5\end{matrix}\right.+\tfrac{1}{2560}\beta^2\left\{\begin{matrix}0\\ Q_5^7\end{matrix}\right.$$

(OEC)
$$\left\{\begin{matrix}\mathfrak{P}_5^4\\ \mathfrak{Q}_5^4\end{matrix}\right.=\tfrac{945}{2}\beta^2\left\{\begin{matrix}P_5\\ Q_5\end{matrix}\right.-18\beta\left\{\begin{matrix}P_5^2\\ Q_5^2\end{matrix}\right.+\left\{\begin{matrix}P_5^4\\ Q_5^4\end{matrix}\right.+\tfrac{1}{40}\beta\left\{\begin{matrix}0\\ Q_5^6\end{matrix}\right.+\tfrac{1}{3840}\beta^2\left\{\begin{matrix}0\\ Q_5^8\end{matrix}\right.$$

(OES)
$$\begin{matrix}\mathsf{P}_5^4\\ \mathsf{Q}_5^4\end{matrix}=\Omega\left[\qquad-9\beta\left\{\begin{matrix}P_5^2\\ Q_5^2\end{matrix}\right.+\left\{\begin{matrix}P_5^4\\ Q_5^4\end{matrix}\right.+\tfrac{3}{80}\beta\left\{\begin{matrix}0\\ Q_5^6\end{matrix}\right.+\tfrac{1}{1920}\beta^2\left\{\begin{matrix}0\\ Q_5^8\end{matrix}\right.\right]$$

(OCC)
$$\left\{\begin{matrix}\mathsf{P}_5^5\\ \mathsf{Q}_5^5\end{matrix}\right.=\Omega\left[\tfrac{63}{4}\beta^2\left\{\begin{matrix}P_5^1\\ Q_5^1\end{matrix}\right.-\tfrac{27}{8}\beta\left\{\begin{matrix}P_5^3\\ Q_5^3\end{matrix}\right.+\left\{\begin{matrix}P_5^5\\ Q_5^5\end{matrix}\right.+\tfrac{7}{240}\beta\left\{\begin{matrix}0\\ Q_5^7\end{matrix}\right.+\tfrac{3}{8960}\beta^2\left\{\begin{matrix}0\\ Q_5^9\end{matrix}\right.\right]$$

(OOS)
$$\left\{\begin{matrix}\mathfrak{P}_5^5\\ \mathfrak{Q}_5^5\end{matrix}\right.=\tfrac{315}{4}\beta^2\left\{\begin{matrix}P_5^1\\ Q_5^1\end{matrix}\right.-\tfrac{45}{8}\beta\left\{\begin{matrix}P_5^3\\ Q_5^3\end{matrix}\right.+\left\{\begin{matrix}P_5^5\\ Q_5^5\end{matrix}\right.+\tfrac{1}{48}\beta\left\{\begin{matrix}0\\ Q_5^7\end{matrix}\right.+\tfrac{1}{5376}\beta^2\left\{\begin{matrix}0\\ Q_5^9\end{matrix}\right..$$

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Note that in this table P_i^s denotes $\dfrac{(\nu^2-1)^{\frac{1}{2}s}}{2^i\ i!}\left(\dfrac{d}{d\nu}\right)^{i+s}(\nu^2-1)^i$, and Ω is $\left(\dfrac{\nu^2-\frac{1+\beta}{1-\beta}}{\nu^2-1}\right)^{\frac{1}{2}}$.

If the variable is μ, and if accordingly the factor $(\nu^2-1)^{\frac{1}{2}s}$ in P_i^s is replaced by $(1-\mu^2)^{\frac{1}{2}s}$, the signs of all the terms which have β as coefficient must be changed. Ω has still the same meaning, but must be written in the form $\left(\dfrac{\frac{1+\beta}{1-\beta}-\mu^2}{1-\mu^2}\right)^{\frac{1}{2}}$.

Table of the Cosine and Sine Functions.

$i=0$ (EEC) $\quad \mathfrak{C}_0=1.$

$i=1$ (OEC) $\quad \mathsf{C}_1=\Phi$

(OOC) (OOS) $\quad \left\{\begin{matrix}\mathfrak{C}_1^1\\ \mathfrak{S}_1^1\end{matrix}\right.=\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right.\phi.$

$i=2$ (EEC) $\quad \mathfrak{C}_2=1-\tfrac{3}{2}\beta\cos 2\phi$

(EOC) (EOS) $\quad \left\{\begin{matrix}\mathsf{C}_2^1\\ \mathsf{S}_2^1\end{matrix}\right.=\Phi\left(\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right.\phi\right)$

(EEC) (EES) $\quad \left\{\begin{matrix}\mathfrak{C}_2^2\\ \mathfrak{S}_2^2\end{matrix}\right.=\tfrac{1}{2}\beta\left\{\begin{matrix}1\\ 0\end{matrix}\right.+\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right.2\phi.$

$i=3$ (OEC) $\quad \mathsf{C}_3=\Phi[1-3\beta\cos 2\phi]$

(OOC) (OOS) $\quad \left\{\begin{matrix}\mathfrak{C}_3^1\\ \mathfrak{S}_3^1\end{matrix}\right.=\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right.\phi-\tfrac{5}{8}\beta(1\pm\tfrac{3}{4}\beta)\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right.3\phi$

(OEC) (OES) $\quad \left\{\begin{matrix}\mathsf{C}_3^2\\ \mathsf{S}_3^2\end{matrix}\right.=\Phi\left[\tfrac{3}{2}\beta\left\{\begin{matrix}1\\ 0\end{matrix}\right.+\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right.2\phi\right]$

(OOC) (OOS) $\quad \left\{\begin{matrix}\mathfrak{C}_3^3\\ \mathfrak{S}_3^3\end{matrix}\right.=\tfrac{3}{8}\beta(1\pm\tfrac{3}{4}\beta)\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right.\phi+\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right.3\phi.$

$i=4$ (EEC) $\quad \mathfrak{C}_4=1-5\beta\cos 2\phi+\tfrac{35}{16}\beta^2\cos 4\phi$

(EOC) (EOS) $\quad \left\{\begin{matrix}\mathsf{C}_4^1\\ \mathsf{S}_4^1\end{matrix}\right.=\Phi\left[\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right.\phi-\tfrac{7}{8}\beta(1\pm\tfrac{5}{4}\beta)\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right.3\phi\right]$

(EEC) (EES) $\quad \left\{\begin{matrix}\mathfrak{C}_4^2\\ \mathfrak{S}_4^2\end{matrix}\right.=\tfrac{9}{4}\beta\left\{\begin{matrix}1\\ 0\end{matrix}\right.+\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right.2\phi-\tfrac{7}{12}\beta\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right.4\phi.$

(EOC) (EOS) $\quad \left\{\begin{matrix}\mathsf{C}_4^3\\ \mathsf{S}_4^3\end{matrix}\right.=\Phi[\tfrac{9}{8}\beta(1\pm\tfrac{5}{4}\beta)\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right.\phi+\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right.3\phi]$

(EEC) (EES) $\quad \left\{\begin{matrix}\mathfrak{C}_4^4\\ \mathfrak{S}_4^4\end{matrix}\right.=\tfrac{3}{16}\beta^2\left\{\begin{matrix}1\\ 0\end{matrix}\right.+\tfrac{1}{3}\beta\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right.2\phi+\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right.4\phi.$

$i=5$ (OEC) $\mathbf{C}_5 = \Phi[1 - 7\beta \cos 2\phi + \frac{63}{16}\beta^2 \cos 4\phi]$

(OOC) (OOS) $\begin{Bmatrix} \mathfrak{C}_5^1 \\ \mathfrak{S}_5^1 \end{Bmatrix} = \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \phi - \frac{7}{4}\beta(1 \pm \frac{15}{8}\beta) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} 3\phi + \frac{21}{32}\beta^2 \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} 5\phi$

(OEC) (OES) $\begin{Bmatrix} \mathbf{C}_5^2 \\ \mathbf{S}_5^2 \end{Bmatrix} = \Phi\left[\frac{15}{4}\beta \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} + \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} 2\phi - \frac{3}{4}\beta \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} 4\phi\right]$

(OOC) (OOS) $\begin{Bmatrix} \mathfrak{C}_5^3 \\ \mathfrak{S}_5^3 \end{Bmatrix} = \frac{3}{2}\beta(1 \pm \frac{15}{8}\beta) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \phi + \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} 3\phi - \frac{9}{16}\beta \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} 5\phi$

(OEC) (OES) $\begin{Bmatrix} \mathbf{C}_5^4 \\ \mathbf{S}_5^4 \end{Bmatrix} = \Phi\left[\frac{15}{16}\beta^2 \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} + \beta \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} 2\phi + \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} 4\phi\right]$

(OOC) (OOS) $\begin{Bmatrix} \mathfrak{C}_5^5 \\ \mathfrak{S}_5^5 \end{Bmatrix} = \frac{5}{32}\beta^2 \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \phi + \frac{5}{16}\beta \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} 3\phi + \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} 5\phi.$

Note that in this table

$$\Phi = (1 - \beta \cos 2\phi)^{\frac{1}{2}}.$$

A table of P (ν) and Q (ν) up to $i = 5$, $s = 5$ is contained in Professor BRYAN's paper (' Proc. Camb. Phil. Soc.,' vol. vi., 1888, p. 297). The functions there tabulated as $T_n^s(\nu)$ and $U_n^s(\nu)$ in the notation here adopted, would be $P_n^s(\nu)$ (with the factor $(\nu^2 - 1)^{\frac{1}{2}s}$) and $(-)^s \frac{i-s!}{i+s!} Q_n^s(\nu)$.

The formula for $Q_i^s(\nu)$, where s is greater than i, is given in § 10 above.